GROUP REPRESENTATIONS AND THE QUANTUM STATISTICS OF SPINS

Jonathan Michael Harrison

School of Mathematics

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Abstract

An intuitive explanation of the connection between the quantum statistics of particles and their spins has been sought since the early proofs of the spin-statistics theorem were published in the 1940's. Recently Berry and Robbins [8] have suggested a new approach. They construct a position-dependent spin basis in which exchanging the positions of identical particles automatically exchanges their spins. In this basis the spin-statistics connection can be derived from the singlevaluedness of the wavefunction. The position-dependent basis for n particles is constructed using the Schwinger representation of spin which can be regarded as a choice of representation of the group $SU(2n)$.

In this thesis we generalise the construction to include all representations of $SU(2n)$. For $n = 2$ vectors that can be used to construct the position-dependent basis are assembled directly using Young tableau and the sign of these vectors under the exchange of the two particles determined. We find that for a typical representations of $SU(4)$ there are several subspaces of vectors with different spins that can be used to construct the position-dependent basis. The sign of vectors in these subspaces under the exchange of the particles is determined not only by the spin but also by the symmetry conditions recorded in the Young tableau which labels the representation of $SU(4)$.

For n particles the decomposition into subspaces that can be used in the construction is achieved using the characters of the relevant groups. We see that typical representations admit parastatistics. The number of subspaces of spin s which transform according to a given irreducible representation of S_n is written in terms of Littlewood-Richardson and Clebsch-Gordan coefficients.

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Author's Declaration

I declare that the work in this thesis was carried out in accordance with the Regulations of the University of Bristol. The work is original except where indicated by special reference in the text and no part of the dissertation has been submitted for any other degree. Any views expressed in the dissertation are those of the author and do not necessarily represent those of the University of Bristol. The thesis has not been presented to any other university for examination either in the United Kingdom or overseas.

Jonathan Michael Harrison

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Chapter 1

Spin-Statistics (*and all that*)

The spin-statistics theorem is a major milestone in the development of quantum mechanics. Its discovery provided an explanation of the Pauli Exclusion Principle which had successfully predicted both the structure of the periodic table and the spectra of atoms. As the Exclusion Principle profoundly changed our view of the world it can be difficult to appreciate how radical the ideas of Pauli and Dirac were in their day. In fact the interest in and controversy around the spin-statistics theorem has never really disappeared since its inception. In this chapter we will follow the development of our current understanding of spin-statistics and the recent attempts to find examples of the violation of the spin-statistics theorem. All this is the physical background to the constructions of non-relativistic spin statistics that will be made in the subsequent chapters.

1.1 The discovery of spin-statistics

The most complete history of the spin-statistics theorem is the book of Duck and Sudarshan [17]. Their treatment reproduces the significant papers along with comments and explanations and is also an enjoyable read. If I can recommend my version it is only that I will be briefer but anyone with an interest in the spin-statistics theorem will benefit from reading their account.

1.1.1 Pauli's exclusion principle

It is the Pauli exclusion principle, a consequence as we will see of the spin-statistics theorem, that was first used to explain the structure of the periodic table and provided a stepping stone in the discovery of the spin of the electron. Stoner realised that by adding an extra "spin" quantum number to label electron states he could explain the $2, 8, 18$ family structure of the periodic table. This inspired Pauli [44] to produce his statement of the Exclusion Principle,

> There can never be two or more equivalent electrons in an atom. These are defined to be electrons for which the value of all quantum numbers is the same. If, in the atom, one electron occurs which has quantum numbers with these specific values, then the state is occupied.

Pauli anticipated a "deeper foundation" for his exclusion principle which would require a better understanding of quantum theory. He continued to pursue these ideas during his research leading eventually to the canonical relativistic argument for the spin-statistics theorem that we have today.

As a consequence of the spin-statistics theorem it is still Pauli's exclusion principle that has the most significant and far reaching implications. Quite literally without this seemingly arbitrary rule for electrons the world as we know it would not exist. The importance of the exclusion principle may explain the calibre of physicists from Heisenberg, Dirac and Fermi to Feynman that contributed to the development of spin-statistics. It certainly explains why Pauli was later awarded the Nobel Prize for his insight.

1.1.2 Spin and statistics

During the discovery of the exclusion principle the additional quantum number attributed to the electron had no physical interpretation. Goudsmit and Uhlenbeck [52] first proposed that this quantum number be assigned to an "eigen-rotation of the electron". They realised that a spherical rotating hollow sphere of charge would have the required gyro-magnetic ratio for spin. The difficulty with this view, that the surface velocity would be greater than the speed of light, was solved later when Dirac introduced the point electron realising the spin as a purely intrinsic property of elementary particles. The electron spin was immediately applied to explain the level splitting in atomic spectra.

With the discovery of spin we have one half of spin-statistics. The statistics we refer to is the statistical mechanics of a gas of identical elementary particles. Initially it was Bose [11] who derived the probability distribution of a photon gas by dividing phase space into cells of volume h^3, where h is Planck's constant. Any number of quanta are assigned to the states of the gas. After corresponding with Bose, Einstein was inspired to extend the ideas to an ideal gas of identical molecules. The Bose-Einstein probability distribution they derive is

$$N_r = \frac{1}{e^{\beta(E_r - \mu)} - 1} \tag{1.1}$$

where

$$\beta \equiv 1/kT$$

and μ is the chemical potential of the gas. N_r is the average occupation number of the state r which has energy E_r. From this probability distribution both the energy spectrum and thermodynamic properties of the gas are then calculated. In the series of papers, [18] [19] [20], the phenomenon of Bose-Einstein condensation was also discovered.

Independently both Fermi [21] and Dirac [15] solved the statistical mechanics of an ideal gas of identical particles which obey the Pauli exclusion principle. Each state is now either occupied by a single particle or is unoccupied. The Fermi-Dirac probability distribution is

$$N_r = \frac{1}{e^{\beta(E_r - \mu)} + 1} \tag{1.2}$$

Again the probability distribution allows the energy spectrum and thermodynamic properties of the gas to be calculated.

The properties of these two ideal gases are clearly very different. At low temperatures a Fermi-Dirac gas fills up all the lowest energy states while a Bose-Einstein gas condenses as the majority of the particles enter the ground state with zero energy and zero momentum. With an understanding of particle spin and statistics we are in a position to state the spin statistics theorem.

> Bosons, particles with Bose-Einstein statistics, all have integer spin while all fermions have half integer spin.

At the time this was a remarkable experimental fact. There are two classes of particles, those that obey or do not obey the Pauli exclusion principle. These two classes possess very different properties. However the membership of the classes is decided by a quantum number, spin, which seems totally unrelated.

1.1.3 The symmetrisation postulate

The symmetrisation postulate was discovered independently by both Dirac [15] and Heisenberg [35] in 1926.

> States containing several identical elementary particles are either symmetric or antisymmetric under permutations of the particles according to the particle species. Bosons are symmetric and fermions are antisymmetric. States which cannot be represented by wave functions with the required symmetry are forbidden.

We will summarise Dirac's argument which appears in the same paper in which he derives the properties of an ideal Fermi-Dirac gas. His paradigm is a system of two electrons orbiting an atom. (mn) denotes the state in which one electron is in the orbit labelled by m and the other in the orbit n. He then asks the question: are (mn) and (nm) two different states?

His argument proceeds as follows. The states (mn) and (nm) are physically indistinguishable. If both states correspond to separate rows or columns in the matrices which operate on the system then the amplitude for the two transitions

$(mn) \rightarrow (m'n')$ and $(nm) \rightarrow (n'm')$ can be individually calculated. They would correspond to two different matrix elements. However as the transitions are physically indistinguishable only the combined intensity should be able to be determined by experiment. So if the theory is to enable only observable quantities to be calculated, (mn) and (nm) must count as a single state (we will see later that this may not necessarily be the case).

Taking the states (mn) and (nm) to be physically indistinguishable has consequences. Only operators that are symmetric in the positions and momenta of the two electrons can be represented by a matrices, as for an operator A

$$A(x_1, x_2)(mn) = A(x_1, x_2)(nm) = A(x_2, x_1)(mn) \tag{1.3}$$

However it is possible to represent the physical properties of the system using matrices which depend symmetrically on the electrons coordinates.

Turning to the eigenfunctions for the two-electron system and neglecting the interaction between the electrons, the eigenfunction for the state (mn) can be constructed from a product of single electron eigenfunctions, $\psi_m(1)\psi_n(2)$. There is, however, a second eigenfunction $\psi_m(2)\psi_n(1)$ which also corresponds to the same state and two independent eigenfunctions would give rise to two rows and columns in the matrices. What is required is a set of eigenfunctions ψ_{mn} of the form

$$\psi_{mn} = a_{mn}\psi_m(1)\psi_n(2) + b_{mn}\psi_m(2)\psi_n(1) \tag{1.4}$$

where the coefficients a_{mn} and b_{mn} are constants. The set should contain only one ψ_{mn} corresponding to the states (mn) and (nm), so applying a permutation ρ of the electrons to the eigenfunction,

$$\rho\,\psi_{mn} = c\,\psi_{mn} \tag{1.5}$$

where c is a phase factor. This set of eigenfunctions must be sufficient to obtain a matrix representation of any operator symmetric in the coordinates, so that

$$A\psi_{mn} = \sum_{m'n'} A_{mn\,m'n'}\,\psi_{m'n'} \tag{1.6}$$

where $\psi_{m'n'}$ is also in the given set of eigenfunctions.

Dirac finds that there are only two solutions, either $a_{mn} = b_{mn}$ or $a_{mn} = -b_{mn}$. Either set of eigenfunctions gives a complete solution of the problem and this choice cannot be determined from the quantum theory. The result extends to any number of electrons, the sets of eigenfunctions are then either symmetric or antisymmetric under permutations of the electrons. As an antisymmetric eigenfunction vanishes for two electrons in the same orbit there can be no more than a single electron in each orbit and we see that the symmetrisation postulate predicts the exclusion principle.

Dirac's argument insists that all states of identical particles be either symmetric or antisymmetric. Later we will be considering parastatistics in which the states of identical particles are allowed not to be entirely symmetric or antisymmetric. As these violate Dirac's result so we should be clear about the essential requirements of his argument. The first condition was that those states corresponding to permutations of the electrons should be physically indistinguishable. This is a strong condition on the states but on its own it is insufficient to deduce the result. The argument also requires the assumption that indistinguishable states are represented by a single vector, up to a phase factor. This was introduced in order to require there to be a single eigenfunction ψ_{mn} corresponding to both states.

Messiah and Greenberg [43] considered the situation where only the indistinguishability condition applies to states. Firstly they find that all physical operators A on the states must be permutation invariant

$$[\rho, A] = 0 \tag{1.7}$$

As the Hamiltonian operator is an observable it also commutes with permutations. Consequently evolving A for a time t we obtain $U^{\dagger}(t)AU(t)$ which is also permutation invariant if A is. They then show that for any vectors in a subspace which transforms according to an irreducible representation of the permutation group the expectation value of $U^{\dagger}(t)AU(t)$ is independent of the particular choice of vector. This is interesting as for more than two particles there are irreducible representations of the permutation group with a dimension of two or more. In this subspace it is then possible to choose two different vectors to represent a state. The choice of vector can't be determined by physical observations.

In the language of Dirac if we considered a three electron state of an atom (lmn) then a wavefunction can be written as a linear combination of all the permutations ρ of the function $\psi_l(1)\psi_m(2)\psi_n(3)$,

$$\psi_{lmn}^{\alpha} = \sum_{\rho} c_{\rho}^{\alpha}\, \psi_l(\rho(1))\psi_m(\rho(2))\psi_n(\rho(3)) \tag{1.8}$$

If the constants c_{ρ}^{α} are chosen such that for a permutation σ

$$\sigma\, \psi_{lmn}^{\alpha} = T(\sigma)_{\beta\alpha}\, \psi_{lmn}^{\beta} \tag{1.9}$$

where $T(\sigma)$ is an irreducible representation of S_3, then ψ_{lmn}^{α} transforms according to the irreducible representation T. This is a generalisation of equation (1.5). In that equation we assumed there could only be a single vector to represent the electron state. If we have selected constants c_{ρ}^{α} so that T is the two dimensional irreducible representation of S_3 then the expectation values of an observable for all ψ_{lmn}^{α} in the two-dimensional subspace will be equal. The indistinguishability of the electrons forces us to choose vectors which belong to irreducible representations of the permutation group but there could still be more than one vector for a given (lmn) in this subspace.

1.2 Relativistic quantum field theory

Quantum field theory was conceived by Dirac [16]. He worked from the canonical commutation relations to define particle creation and annihilation operators. These operators add or remove quanta from states which can be multiply occupied and so the theory is therefore a field theory of bosons. In order to define a field theory for fermions Jordan and Wigner [39] replaced the commutation relations for the creation and annihilation operators with anticommutation relations. Using these operators they define antisymmetric states and wavefunctions which obey the Pauli exclusion principle. So in field theory the spin-statistics relation becomes a connection between the choice of commutators or anticommutators for the creation and annihilation operators of the field and the spin of the particles represented by the field.

1.2.1 Pauli's proofs

While the original field theories were non-relativistic, by quantising the Klein-Gordon equation Pauli and Weisskopf [46] produced the canonical relativistic quantum field theory. Pauli then used this relativistic quantum field theory to attempt his first proof of the spin-statistics theorem. The idea was to show that the relation between spin and statistics is a necessary consequence of the postulates of relativistic quantum field theory. In this respect all the field theory proofs are alike although they differ depending on the precise axioms of the field theory used.

Pauli's first proof was not conclusive. He himself questioned the validity of several of the operations he used and it was some years before he reached what is now regarded as his orthodox proof of the spin-statistics relation. While Pauli began his work on the spin-statistics relation Iwanenko and Socolow [38] quantised the Dirac equation using anticommuting creation and annihilation operators. They concluded that attempting to apply Bose statistics to the Dirac equation inevitably produces problems and so Bose statistics are most natural for the scalar relativistic equation while Fermi statistics are natural for Dirac's relativistic equation.

Before discussing Pauli's second proof we should note some of the significant ideas contributed by less recognised authors. Fierz [25] introduced the notion of representing elementary particles with irreducible relativistic spinors which Pauli would later generalise. Belinfante's unique approach [7] was to require invariance under the charge-conjugation transformation. This was not only novel but, interestingly, the argument is now used in reverse, the proof of the spin-statistics theorem being the foundation for the proof of the PCT theorem. DeWet [14] also produced a proof based on canonical field theory in which he was the first to identify one of the crucial assumptions on which many proofs from relativistic field theory depend,

$$[\phi(x), \phi^\dagger(y)]_+ \geq 0 \quad \text{for } (x-y) \text{ spacelike}$$

where ϕ is a field operator. It would be fifteen years before this was proved.

That Pauli's name is so strongly linked to the spin-statistics theorem is due to his 1940 proof [45] based on a classification of the spinor representations of the proper Lorentz group. Although the proof relies on field theory we can see the origin of the spin-statistics connection in the representations of the spinors. The proper Lorentz group is the continuous group of linear transformations which leaves invariant the scalar product

$$\sum_{k=0}^{3} x^k x_k = x_1^2 + x_2^2 + x_3^2 - x_0^2 \tag{1.10}$$

Pauli uses the spinor representations of the Lorentz group. A basic spinor is defined from a four-vector $\mathbf{v}$ by the relation

$$U^{\alpha\dot{\beta}} = v^\mu \sigma_\mu^{\alpha\dot{\beta}} \tag{1.11}$$

σ_μ are the Pauli matrices with σ_0 defined to be the identity matrix. The group multiplication law is then the normal matrix multiplication for the spinor matrices. The inverse relation to return a four-vector from a spinor is given by

$$v^\mu = -\frac{1}{2}\sigma^\mu_{\alpha\dot{\beta}} U^{\alpha\dot{\beta}} \tag{1.12}$$

Spinor indices are raised and lowered with the alternating tensor $\epsilon^{\alpha\beta} = -\epsilon^{\beta\alpha}$ with $\epsilon^{12} = \epsilon_{12} = 1$ etc. A general spinor $U^{\dot{\mu}\dot{\nu}\dot{\sigma}\ldots}_{\alpha\beta\gamma\ldots}$ can be characterised by two "angular momentum quantum numbers" (j,k) where there are $2j$ upper dotted indices and $2k$ lower indices.

Pauli divides the representations into classes depending on their properties when the representation is restricted to the subgroup of space rotations. If we take a representation $U(j,k)$ where $j+k$ is half integral, then applying a space rotation by 2π vectors in the representation space undergo a sign change. In a representation where

$j+k$ is integral vectors in the representation space don't change sign under a 2π rotation. Pauli refers to these representations as single- or double-valued. The spinor representations U which transform as double-valued representations of the Lorentz group correspond to particles with half-integral spin, while the single-valued representations correspond to those with integral spin.

The single-valued representations are further classified into representations where j and k are both integral, U^+, and those where both j and k are half integral, U^-. The direct product of two representations decomposes into a sum of irreducible representations in a particular class,

$$\begin{gathered} U^+U^+ = U^+ \qquad U^-U^- = U^+ \\ U^+U^- = U^-U^+ = U^- \end{gathered} \tag{1.13}$$

For the double valued representations $U^{+\epsilon}$ refers to representations where j is integral and k half integral and $U^{-\epsilon}$ to j half integral k integral. The multiplication table for these representations is

$$\begin{gathered} U^{\pm\epsilon}U^{\pm\epsilon} = U^+ \qquad U^{\pm\epsilon}U^{\mp\epsilon} = U^- \\ U^{\pm\epsilon}U^+ = U^{\pm\epsilon} \qquad U^{\pm\epsilon}U^- = U^{\mp\epsilon} \end{gathered} \tag{1.14}$$

Pauli considers both commutation and anticommutation relations for the four classes of spinor fields. In either case he postulates the brackets of the field operators can be expressed in terms of an invariant D-function and the derivatives of that function,

$$[U(\mathbf{x}), \overline{U}(\mathbf{x}')]_\pm \sim D(\mathbf{x}-\mathbf{x}') \tag{1.15}$$

where $\pm$ on the bracket refers to anticommutation or commutation relations respectively and $\overline{U}$ is the complex conjugate of U. The D-function is given by

$$D(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}.\vec{x}} \frac{\sin \omega x_0}{\omega} \tag{1.16}$$

$\vec{x}$ is the normal three vector position and similarly $\vec{p}$ is the momentum. The D-function is uniquely determined by the conditions,

$$(\Box - m^2)D = 0 \qquad D(\vec{x},0) = 0 \qquad \partial_0 D|_{x_0=0} = \delta(\vec{x}) \tag{1.17}$$

Using the rules for multiplying the spinor representations, one can find conditions on the brackets. For half-integral spins with either commutation or anticommutation relations

$$[U^{\pm\epsilon}, \overline{U}^{\pm\epsilon}]_{\pm} = [U^{\pm\epsilon}, U^{\mp\epsilon}]_{\pm} = U^{-} \tag{1.18}$$

To construct a spinor in U^{-} from D and its derivatives only the odd derivatives of D can appear. For integral spins the brackets must have the opposite form

$$[U^{\pm}, \overline{U}^{\pm}]_{\pm} = [U^{\pm}, U^{\pm}]_{\pm} = U^{+} \tag{1.19}$$

These brackets correspond to only even derivatives of D. Pauli then symmetrises these relations for permutations of the positions, $\mathbf{x} \leftrightarrow \mathbf{x}'$.

$$X = [U^{\pm}(\mathbf{x}), \overline{U}^{\pm}(\mathbf{x}')]_{\pm} + [U^{\pm}(\mathbf{x}'), \overline{U}^{\pm}(\mathbf{x})]_{\pm} \tag{1.20}$$

X is therefore even under $\mathbf{x} \leftrightarrow \mathbf{x}'$. He shows that the odd derivatives of the D-function are even under a permutation of the positions, $\vec{x} \leftrightarrow \vec{x}'$ but odd under $x_0 \leftrightarrow x_0'$. Consequently for integral spins the X vanishes under symmetrisation. This excludes the possibility of anticommutation relations for integral spin particles because at $x = x'$ the expression for X would be positive and could therefore only vanish for fields which are identically zero.

For half integral spins the derivatives of the D-function are even under the permutation $\mathbf{x} \leftrightarrow \mathbf{x}'$. To rule out commutation relations Pauli uses an argument due to Fierz that the energy of the system is only positive when anticommutation relations are chosen. In this way he establishes the connection between spin and particle statistics without it being necessary to fix the particular spin of the particles in question. There are still problems with the proof; as with all proofs of the time the manipulations of the field are not shown to be valid and interactions are not included. However it was a great advance and also includes a discussion of symmetry conditions which anticipated the PCT theorem but which I have omitted from this outline.

The aspect of Pauli's proof of most interest in this thesis is his use of double-valued representations. When we discuss the construction of Berry and Robbins in

Chapter 3 we will also encounter double-valued representations where, under restriction to a subgroup of permutations, vectors of half-integral spin change sign. The group used is however not the Lorentz group but the group of special unitary matrices. The analogy is very weak but at least in a historical context it is an interesting connection between the two approaches.

1.2.2 Axiomatic proofs

While Pauli is widely credited with the derivation of the spin-statistics theorem the most complete proofs are due to Lűders, Zumino [51] and Burgoyne [13]. The most memorable reference for a proof of the spin-statistics theorem is however the book of Streater and Wightman [50]. Their proof also follows the axiomatic approach which Lűders, Zumino and Burgoyne introduced.

Before these axiomatic proofs were developed there were two other contributions to the history of the spin-statistics theorem that should be noted. In 1949 Feynman [23] published a paper purporting to show that only the observed spin-statistics connection was compatible with calculations of the vacuum survival probability made using the Feynman rules. The approach was very novel but an analysis by Pauli showed the theory required an indefinite metric on the Hilbert space while a basic postulate of the field theory is that the metric is positive definite. Undeterred this served as a basis for Feynman's ideas for an elementary proof. Schwinger [49] published a proof based on the requirement that relativistic quantum field theory be time reversal invariant. In the later proof of Lűders and Zumino and also in Streater and Wightman's book this is reversed so that both the spin-statistics theorem and PCT theorem rest on the basic axioms of relativistic field theory.

In their book Streater and Wightman follow the proof of Burgoyne. The work of Lűders and Zumino applies only to spin 0 and 1/2 although it has the advantage of clearly separating the PCT and spin-statistics theorems. Both Lűders and Zumino and Streater and Wightman make use of the Hall-Wightman theorem. As this theorem is often used in proofs of the spin-statistics connection using relativistic quantum field theory I will state it here.

Theorem: Hall-Wightman 1.2.1. *The vacuum expectation value of the product of two fields*

$$\langle 0|\phi(\mathbf{x})\phi(\mathbf{x}')|0\rangle = F(\mathbf{x}-\mathbf{x}')$$

is analytic in $(\mathbf{x}-\mathbf{x}')$ *and continuable to all separations.*

In Streater and Wightman the discussion of such results extends over most of a chapter, including it here is only designed to give a flavour of the problems to be tackled in order to produce rigorous proofs.

As well as having a full understanding of the allowed manipulations of field operators the other essential building block of these proofs are the axioms of relativistic quantum field theory. The mathematical discussion of Streater and Wightman is too involved for an introduction so instead I will refer to the original postulates of Burgoyne.

1. The field theory must be relativistically invariant.
2. The theory contains no negative energy states. (This is equivalent to requiring the vacuum state to be the lowest energy state.)
3. The metric in Hilbert space is positive definite.
4. Distinct fields either commute or anticommute for space-like separations.

He then shows that for any field with these properties the "wrong" connection between spin and statistics implies that the field vanishes. For readers with some knowledge of field theory the proof is structured as follows;

From field operators $\Phi_\mu(\mathbf{x})$, which transform according to an irreducible representation of the homogeneous Lorentz group indexed by μ, we define tempered field operators

$$\Phi(f) = \int d^4x\, f^\mu(\mathbf{x})\Phi_\mu(\mathbf{x}) \tag{1.21}$$

where the $f^\mu(\mathbf{x})$ are a set of test functions. F and G are defined to be the vacuum expectation values

$$F_{\mu\lambda}(\boldsymbol{\xi}) = \langle 0|\Phi_\mu(\mathbf{x})\overline{\Phi}_\lambda(\mathbf{x}')|0\rangle \tag{1.22}$$

$$G_{\mu\lambda}(\boldsymbol{\xi}) = \langle 0|\overline{\Phi}_\mu(\mathbf{x})\Phi_\lambda(\mathbf{x}')|0\rangle \tag{1.23}$$

with $\boldsymbol{\xi}$ the relative position $(\mathbf{x}-\mathbf{x}')$. F and G can be extended to functions of a complex four vector $\mathbf{z} = \boldsymbol{\xi} - i\boldsymbol{\eta}$ which are analytic for $\mathbf{z}^2 = \mathbf{z}^j\mathbf{z}_j$ in the complex plane cut along the positive real axis. Burgoyne shows that for space like separations $\boldsymbol{\xi}^2 < 0$,

$$G_{\mu\lambda}(-\boldsymbol{\xi}) = (-1)^{2s} G_{\mu\lambda}(\boldsymbol{\xi}) \tag{1.24}$$

where s is the particle spin. The "wrong" sign commutation relations for the field operators at space-like separations are

$$\langle 0|\Phi_\mu(\mathbf{x})\overline{\Phi}_\lambda(\mathbf{x}') + (-1)^{2s}\overline{\Phi}_\lambda(\mathbf{x}')\Phi_\mu(\mathbf{x})|0\rangle = 0 \tag{1.25}$$

We have used the fourth axiom of the field theory that the fields commute or anti-commute for space-like separations. Equation (1.25) implies that

$$F_{\mu\lambda}(\boldsymbol{\xi}) + (-1)^{2s} G_{\lambda\mu}(-\boldsymbol{\xi}) = 0 \tag{1.26}$$

Using (1.24) we see that

$$F_{\mu\lambda}(\boldsymbol{\xi}) + G_{\lambda\mu}(\boldsymbol{\xi}) = 0 \tag{1.27}$$

By analyticity $F_{\mu\lambda} + G_{\lambda\mu}$ vanishes everywhere in the cut $\boldsymbol{\xi}$ plane. To evaluate the limit $\boldsymbol{\xi} \to 0$ the tempered fields are used. From equation (1.26)

$$\langle 0|\Phi(f)\overline{\Phi}(f) + \overline{\Phi}(g)\Phi(g)|0\rangle = 0 \tag{1.28}$$

As the Hilbert space metric is positive definite we conclude that

$$|\overline{\Phi}(f)|0\rangle|^2 + |\Phi(g)|0\rangle|^2 = 0 \tag{1.29}$$

Consequently the tempered field operators $\overline{\Phi}(f)$ and $\Phi(g)$ are identically zero for all test functions f and g. The field is therefore zero and the "wrong" commutation relations are untenable.

The canonical proof of the spin statistics theorem given by Streater and Wightman is a combination of the proof of Burgoyne with the spinor proof of Pauli, the spinors being used to prove the equivalent of statement (1.24). It has the advantage over the Pauli proof of being rigorous with theorems for all the necessary manipulations of the field operators also proved. It is probably for this reason that amongst such a wide range of approaches it has achieved the status of the definitive proof.

1.2.3 Criticisms of the field theory proofs

In the outlines of the proofs of the spin-statistics theorem from relativistic quantum field theory I hope I have been fair both to their achievements and to the degree of the formalism that they necessarily introduce. The axiomatic approach of Streater and Wightman can appear to reduce the spin-statistics relation to a mathematical problem involving the existence of an analytic continuation of certain tempered distributions in convex cones of four-dimensional space time. If there is a physical result obscured by this analysis the best candidate is the connection between spin and the double- or single-valued representations of the proper Lorentz group. This is not however used in all the proofs so it is hard to see it as a physical basis for the spin-statistics connection. If from these relativistic arguments we were to try and explain the spin-statistics theorem what we can say is that another connection between spin and statistics is incompatible with the axioms of relativistic quantum field theory and we must be satisfied with that.

1.3 Feynman and the elementary proofs

In Feynman's lectures on physics [22] he discusses the spin-statistics connection and asks the following question:

> "Why is it that particles with half-integral spin are Fermi particles whereas particles with integral spin are Bose particles? We apologise

> for the fact that we cannot give you an elementary explanation. An explanation has been worked out by Pauli from complicated arguments of quantum field theory and relativity. He has shown that the two must necessarily go together, but we have not been able to find a way of reproducing his arguments on an elementary level. It appears to be one of the few places in physics where there is a rule which can be stated very simply, but for which no one has found a simple and easy explanation. This probably means that we do not have a complete understanding of the fundamental principle involved."

This question has inspired several attempts to formulate such an elementary argument and it is in this spirit that Michael Berry and Jonathan Robbins proposed their non-relativistic construction of the spin-statistics connection with which the rest of this thesis is concerned. Here we will review other approaches that have been taken to the question. The main critiques of these proposals have been provided by Hilborn [36] and Sudarshan and Duck [17].

1.3.1 Geometric rotation

Probably the most well known of the "elementary" schemes are the geometric arguments of Bacry [4] and Broyles [12]. The simple situation described by Bacry is sufficient to demonstrate the essential idea. Unfortunately both arguments contain the same critical flaw.

Bacry considers the two electron system where one electron is located at coordinates $(x, y, z) = (a, 0, 0)$ with spin $+1/2$ and the second electron at $(-a, 0, 0)$ with spin $-1/2$. The single particle wavefunctions are

$$\psi_A = \begin{pmatrix} \delta(x-a)\delta(y)\delta(z) \\ 0 \end{pmatrix} \qquad \psi_B = \begin{pmatrix} 0 \\ \delta(x+a)\delta(y)\delta(z) \end{pmatrix} \tag{1.30}$$

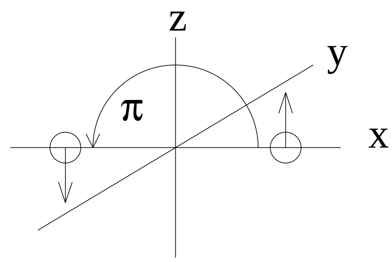
and the two electron wavefunction is written

$$\Psi_{AB}(1,2) = \psi_A(1)\psi_B(2) \pm \psi_A(2)\psi_B(1) \tag{1.31}$$

We have not as yet chosen whether the wavefunction for the system will be symmetric or antisymmetric. Exchanging the two particles

$$\Psi_{AB}(2,1) = \pm\Psi_{AB}(1,2) \tag{1.32}$$

A rotation by π about the y axis leaves the two electron state unchanged.



If we act on the single electron wavefunctions with the rotation about the y axis $R_y(\pi) = e^{-i\pi J_y}$

$$R_y(\pi)\psi_A = \psi_B \qquad R_y(\pi)\psi_B = -\psi_A \tag{1.33}$$

So acting on the two electron wavefunction

$$R_y(\pi)\Psi_{AB}(1,2) = - \pm \Psi_{AB}(1,2) \tag{1.34}$$

Bacry then makes an error which invalidates the proof, the proof of Broyles also fails for a similar reason. As the two electron state is invariant under the rotation $R_y(\pi)$ he requires the same of the wavefunction

$$R_y(\pi)\Psi_{AB}(1,2) = \Psi_{AB}(1,2) \tag{1.35}$$

If this were the case it would require the wavefunction to be antisymmetric which is the spin-statistics relation.

Unfortunately the invariance of the state under a discrete symmetry transformation does not rule out the possibility of a sign change in the wavefunction. An example of the phenomena is a 2π rotation of a spin-1/2 state of an electron. The state is invariant but the wavefunction changes sign. As the sign of $\Psi_{A,B}(1,2)$ under the rotation $R_y(\pi)$ cannot be determined the argument fails to provide a spin statistics connection.

1.3.2 Feynman's Dirac lecture

In 1986 Feynman, despite his long illness, gave the Dirac memorial lecture in which he sketched an elementary argument for the spin-statistics connection [24]. He was inspired by the unexpected behaviour of tethered classical objects under rotations. The lecture demonstration was rotating a full wine glass through 4π with out spilling the water. The same idea is contained in a trick with a belt which is easier to describe. If a belt is fixed at one end while the other end is connected to an object rotating the object by 2π introduces a twist to the belt which cannot be undone by translating the object. However the twists in the belt that result from rotating the object by 4π can be removed by translating the object whilst keeping its orientation fixed. Figure 1.1 shows a schematic of the belt trick though it is best verified personally. In the figure a straight belt is rotated through 2π then 4π. Finally keeping the orientation of the black cube fixed whilst translating along the path shown removes the twists in the belt.

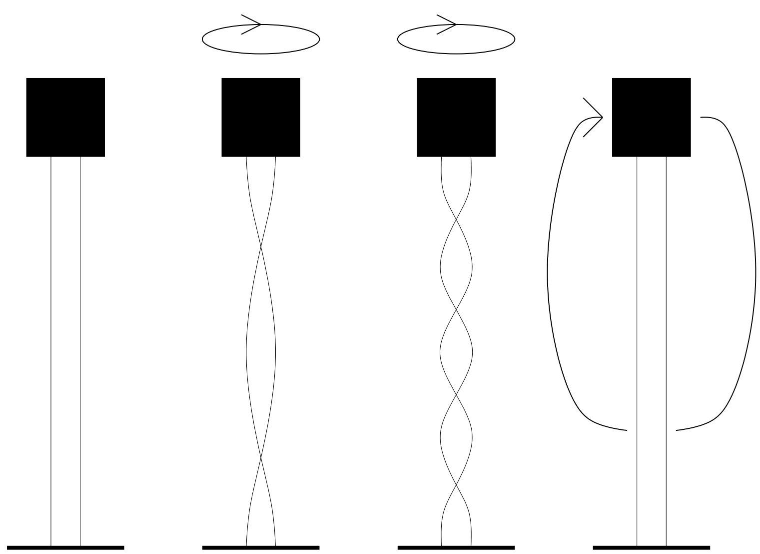


Figure 1.1: The First Belt Trick

This classical paradox suggests that a rotation by 2π does not always return an object to its original state and therefore that the change in sign of the wavefunction of spin-1/2 particles is not unreasonable. In their discussion of these ideas Sudarshan and Duck warn the unwary not to be deceived by such party tricks. In their words they are "pure old fashioned snake-oil peddling". However they concede that in Feynman's hands they were mesmerising.

With this model Feynman also included two others in which the exchange of identical particles reproduces the spin-statistics connection. One model uses a pair of composite particles each consisting of a spin zero electric charge e and a spin zero magnetic monopole of charge g. However as an explanation of spin-statistics requiring elementary particles to have the additional unphysical property of sourcing a magnetic field is not compelling.

The third model makes use of a second belt trick, see figure 1.2. In the trick when two particles connected by a ribbon are exchanged the ribbon acquires a twist. To complete the exchange one of the particles must be rotated by 2π. The argument is used to demonstrate that the operation of exchange entails a hidden rotation by 2π. Feynman uses this rotation to determine the effect of exchange on particle states from which he acquires the spin-statistics relation. The reasonable objection raised by Duck and Sudarshan is that we are required to postulate that elementary particles are connected by ribbons a property which is needed for no other purpose. While the classical argument suggests a spin-statistics like result a derivation of the spin-statistics theorem should be based on more natural physical properties.

In his lecture Feynman also returned to his original field theory argument. He used the time reversal property of the Dirac spinor to show that Fermi-Dirac statistics are necessary if the S matrix is to be unitary. This use of the PCT theorem to prove the spin-statistics theorem reverses the status of the theorems and requires that the PCT theorem be proved without the use of a spin-statistics relation. In their analysis of Feynman's argument Sudarshan and Duck conclude that the internal consistency of the Feynman rules for perturbation theory is not a sufficient foundation for the spin-statistics theorem.

Feynman's comments on the second belt trick were inspired by a rigorous ge-

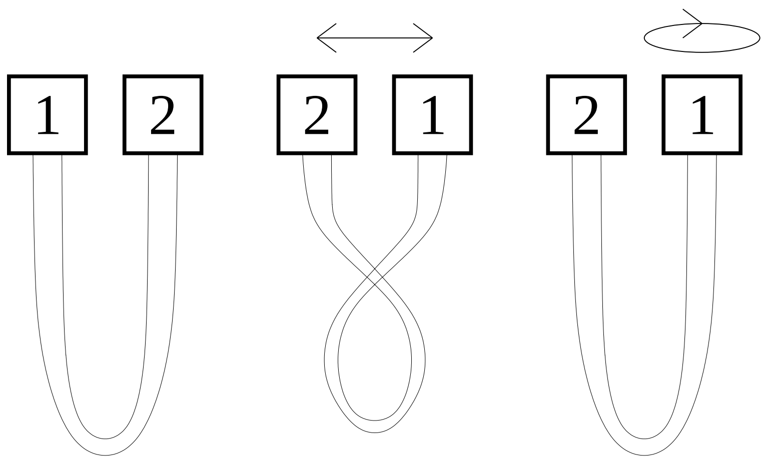


Figure 1.2: The Second Belt Trick

ometric argument of Finkelstein and Rubinstein [26]. Finkelstein and Rubinstein treat nonlinear field theories in which there exist modes of the field, called kinks or solitons, which can not be deformed into each other and which posses a conserved integer particle number. They define an exchange operation in which two soliton antisoliton pairs are created, the solitons exchanged, and the new pairs annihilated. The eigenvalues of the exchange operator are ± 1 corresponding to even or odd statistics. This exchange operation is shown to be homotopic to a 2π rotation of the field which implies solitons with half integer spin have odd statistics. Those soliton species which can undergo odd exchanges also admit even exchanges. To fix the statistics and so the spin of a given type of soliton they note that all solitons of the same type must produce the same sign under exchange and the sign can not change over time. Consequently given a universe of solitons measuring the exchange sign for a pair fixes the exchange property of all solitons of that type for all time. Parastatistics, which will be discussed later, is excluded in this model.

1.3.3 Recent candidates for an elementary proof

Balacharandran et al [6] (see also [5] for a brief account) have suggested a development of the argument of Finkelstein and Rubinstein applied to point particles. Their proof avoids reference to field theory or relativity. However the argument makes use of an infinite dimensional configuration space. The configuration space of a single particle is $\mathbb{R}^3 \times F^3(SO(3))$ where $SO(3)$ is the group of rotations and F^3 is the set of all orthonormal frames with a fixed orientation in 3 dimensions, this determines the particles spin. Antiparticles are described by a state in $\mathbb{R}^3 \times \overline{F}^3(SO(3))$ where the orthonormal frames in $\overline{F}^3$ have the opposite orientation to the particle frames. For states of many identical particles the configurations where spins and positions have been exchanged are identified and pairs of particles are not allowed to occupy the same position. The effect of particle antiparticle annihilation is included by associating configurations where particles and antiparticles coincide with the configuration space of reduced particle numbers. Assuming certain continuity conditions on this configuration space they show that exchange for particles with spin is homotopic to a 2π rotation of one of the particles which is the spin-statistics connection. Establishing this homotopy makes use of the creation and annihilation of particle antiparticle pairs. They suggest that the analogue of these arguments in field theory will involve the physics of solitons although the question is not resolved.

Sudarshan and Duck in the final chapter of their book [17] include their own elementary argument. They replace the postulates of relativistic quantum field theory with conditions on the kinematic parts of the Lagrangian for the individual fields. The Lagrangian must be

1. derived from a local Lorentz invariant field theory for fields which are each a finite dimensional representation of the Lorentz group.
2. in the Hermitian field basis.
3. at most linear in the first derivatives of the fields.
4. at most bilinear in the fields.

They then show that the "wrong" spin-statistics connection is incompatible with rotational invariance of the Lagrangian.

This proof also seems to fail to provide the sought after elementary physical understanding of the spin-statistics connection that Feynman requested. In their discussion of their result Duck and Sudarshan say the simplification they introduce is the use of rotation invariance rather than time reversal invariance in Schwinger's field theory proof. However the essential structure remains the same, they show that only the observed spin-statistics connection is compatible with a given set of requirements for the field as in the axiomatic proofs. Although they claim the proof is non-relativistic the requirement that the field be Lorentz invariant is still retained.

In summary the search for an elementary proof of the spin-statistics connection has produced many interesting and significant analogues of the spin-statistics connection. Unfortunately none are conclusive. Some require additional unphysical properties to be postulated for elementary particles while others remain refinements of the relativistic argument.

1.4 Parastatistics

A good review of the theories which allow a violation of the spin-statistics connection is provided by Greenberg [31]. Here we will tackle only a couple of the topics namely, Green's parastatistics, which is the original example of parastatistics, and the theory of quons, which has particular relevance to experimental attempts to look for small violations in particle statistics.

1.4.1 Parabosons and Parafermions

Parastatistics, proposed by Green [30], is a theoretical generalisation of Bose and Fermi statistics. Bose or Fermi statistics are defined by the choice of commutation or anticommutation relations for the creation and annihilation operators of particle states. The number operator for a state k can be written

$$n_k = \frac{1}{2}[a_k^\dagger, a_k]_\pm + \text{ const.} \qquad (1.36)$$

where $+$ on the bracket refers to the choice of anticommutation relations and $-$ to commutation relations for the operators a. The operator $a_k^\dagger$ creates a particle in the state k and a_k is the corresponding annihilation operator. The bracket is either a commutator or an anticommutator according to the type of statistics. The commutator of the number operator and creation operators is independent of the choice of Bose or Fermi statistics.

$$[n_k, a_l^\dagger]_- = \delta_{kl} a_l^\dagger \tag{1.37}$$

We can generalise the definition of the number operator to use the operators $a_k^\dagger$ and a_m annihilating a particle in state m and creating one in state k. Then substituting this number operator into equation (1.37) we find Green's trilinear commutation relations.

$$[[a_k^\dagger, a_m]_\pm, a_l]_- = 2\delta_{ml} a_k^\dagger \tag{1.38}$$

Selecting the commutator or anticommutator in this relation will define two alternative cases of *parabose* or *parafermi* statistics. As these commutation rules are trilinear the definition of the vacuum state

$$a_k|0\rangle = 0 \tag{1.39}$$

is insufficient to allow all states to be calculated. An additional condition on single-particle states is included to resolve this

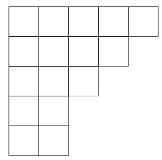
$$a_k a_l^\dagger |0\rangle = p\, \delta_{kl} |0\rangle \tag{1.40}$$

To find solutions of these commutation rules Green made the following ansatz. Let

$$a_k^\dagger = \sum_{\alpha=1}^{p} b_k^{(\alpha)\dagger} \qquad a_k = \sum_{\alpha=1}^{p} b_k^{(\alpha)} \tag{1.41}$$

The parabose solutions are defined by taking the pair of operators $b_k^{(\alpha)}$, $b_k^{(\beta)}$ to commute if $\alpha = \beta$ but anticommute if $\alpha \neq \beta$. The parafermi statistics are found by swapping the use of commutation and anticommutation relations in the definition. The ansatz provides a set of parabose and parafermi statistics for each integer p. For parabosons p is the maximum number of particles which can occupy an antisymmetric state, while for parafermions it is the maximum number of particles which can occupy a symmetric state. These parastatistics were the first alternative to the observed statistics of Fermi-Dirac or Bose-Einstein to be defined.

The irreducible representations of the symmetric group are labelled by Young tableau, discussed in greater detail in chapter 2.



We will see later that the tableau record symmetry conditions. Single particle states are assigned to boxes in the tableau then a tensor product of n single particle states is symmetrised with respect to the states in the same row then antisymmetrised with respect to those in the same column. The parabose or parafermi systems transform according to irreducible representations of the symmetric group labelled by tableau with no more than p rows or columns respectively. These correspond to having at most p particles in antisymmetric or symmetric states. There are other forms of parastatistics in which all representations of the symmetric group are admissible. States with the original Fermi statistics are represented by a tableau with a single column, the state is purely antisymmetric. Equivalently Boson states are represented by tableau with a single row where the state is completely symmetric. We see now that the definitions of parabosons and parafermions are particular examples of operators whose states transform according to more complex representations of the symmetric group. The term parastatistics is used to cover all systems in which particle states transform according to these generalised symmetry conditions.

1.4.2 Quons

In most experiments which look for a violation of the spin-statistics theorem the assumption is that the expected statistics will be violated by a small amount. To compare these experiments to a theory requires a model in which the statistics can vary with a small parameter q. Bounds on the size of q then give a quantitative measure of the accuracy of the spin-statistics theorem. To date the best theoretical framework for such a violation of particle statistics is the quon.

The quon algebra is defined by taking the convex sum of the Bose and Fermi algebras.

$$\frac{1+q}{2}[a_k, a_l^\dagger]_- + \frac{1-q}{2}[a_k, a_l^\dagger]_+ = \delta_{kl} \qquad (1.42)$$

q is in the range $-1 \leq q \leq 1$. The usual vacuum condition

$$a_k|0\rangle = 0 \qquad (1.43)$$

is sufficient to calculate matrix elements of polynomials in the creation and annihilation operators. At $q = 1$ the statistics are bosonic while at $q = -1$ they are fermionic. Given the discrete representations of the symmetric group we should be clear about the sense in which q interpolates between these statistics. Vectors formed by polynomials in the creation operators acting on the vacuum are superpositions of vectors in different irreducible representations of the symmetric group. As we vary q the weight given to vectors in these irreducible representations varies smoothly. For example as $q \to 1$ the weights of all representations tend to zero with the exception of the symmetric representation leaving a completely symmetric state.

The quon theory possesses many of the desired properties for a theory which allows small violations of spin statistics. The norms are positive, cluster decomposition theorems and the PCT theorem hold and free fields can have relativistic kinematics. It is not however ideal as observables with space-like separations don't commute and as a consequence interacting relativistic field theory may not be possible.

1.5 Experimental tests

To conclude this introduction to the spin-statistics theorem we will consider the current experimental evidence for the symmetrisation postulate. Gillaspy [28] provided a review of the literature from which most of our data will be taken.

Currently there are no examples of particle behaviour which violate the symmetrisation postulate. If in fact such data were to be seen, for example an inhibited transition in a collider experiment, it is unlikely it could be attributed to such a violation. The frequency of such violations would be so low that the possibility of an error in the experiment or detectors would prevent our gedanken investigator publishing. Instead experiments to test the symmetrisation postulate yield upper bounds on the probability β^2 of finding a two particle state with unusual statistics. In terms of the quon model of particle statistics

$$\beta^2 = \begin{cases} 1+q & \text{for fermions} \\ 1-q & \text{for bosons} \end{cases} \tag{1.44}$$

The limits put on β^2 by experiments vary widely. Gillaspy attributes this to the trade-off between the chance of the experiment observing a violation and the size of the violation that it is capable of detecting.

Before discussing particular types of experiments we should mention some of the general problems encountered when attempting to verify such a fundamental law of physics. As elementary matter particles are fermions few experiments in to the symmetrisation postulate are carried out on bosons. There are however approximate results which suggest that the scale of a violation in the two classes should be comparable. Using the indistinguishability of identical particles it can be shown that it is not possible for a particle in an ordinary state to transfer to a state violating the symmetrisation postulate. This important superselection rule prevents many symmetry violating transitions. For example we might assume that if there were a small violation of the symmetrisation postulate then electrons in a block of matter would slowly relax into the ground state, emitting photons in the process. This transfer is inhibited by the indistinguishability of electrons and so the absence of such transitions does not provide a test of the symmetrisation postulate.

We can now consider some of those schemes that have been used to provide bounds on β^2.

Absorbing blocks: Fresh electrons which could be in a symmetry violating state are absorbed by the block. They are able to bind to an excited state of the atoms and might then decay to the ground state emitting a photon. The most precise experiment of this type was conducted by Ramberg and Shaw yielding $\beta^2 \lesssim 10^{-26}$.

Decaying blocks: A nuclear reaction in the block ejects a fresh particle from the block. Associated with the ejected particle is another that could have anomalous symmetry and decay to the ground state. The most precise data for an experiment of this type provides a limit of $\beta^2 \lesssim 10^{-57}$.

Collisions in vacuum: Individual particles are projected into atoms in a vacuum. The separate results can then be analysed. As there are few events the system is less sensitive but hopefully more accurate. $\beta^2 \lesssim 10^{-13}$.

Ground state accumulation: In a typical experiment mass spectroscopy is used to search for atoms with an anomalous number of electrons in their ground state. Whether the chemical behaviour of such an anomalous atom would remain unchanged remains an open question. A current limit on violation from this type of experiment is $\beta^2 \lesssim 10^{-27}$.

The limits from these experiments depend to a great extent on the assumptions made in their analysis. Interestingly those experiments producing the lowest bounds are not necessarily the most recent. These experiments and results are for small violations of the symmetrisation postulate of the type we might expect from the quon theory. An alternative experimental consideration for the symmetrisation postulate are the statistics of the more exotic elementary particles. For these particles with short lifetimes it is difficult to first create and then make measurements on two particle states. While for some particles like the pion the spin-statistics theorem has been confirmed there are many elementary particles for which successful experiments have yet to be devised.

1.6 Conclusions

The aim of this introduction to the spin-statistics theorem has been to summarise the present understanding of spin-statistics. We have seen how the structure of the periodic table and atomic spectra led to the discovery of Pauli's exclusion principle and the symmetrisation postulate. These core properties of nature are not a consequence of quantum mechanics, which does not specify the symmetry properties of wavefunctions. The search for a theoretical explanation of these phenomena then centred around relativistic quantum field theory where the most complete modern proofs show that integer spin particles with Fermi-Dirac statistics or half integer spin particles with Bose-Einstein statistics are not consistent with the basic axioms of the field theory. These proofs are however negative proofs and we have seen that parastatistics in which wavefunctions transform according to any irreducible representation of the permutation group are also consistent with quantum mechanics even though they have not been observed by experiment. There also exists a local algebra approach to quantum field theory in which the algebras of bounded operators generated by observables in bounded regions of space-time are studied. In this theory the various kinds of parastatistics appear under appropriate conditions, see Haag [32].

A proof of the spin-statistics theorem would be more satisfactory if it were to be derived from a clear physical principle which allowed a more intuitive understanding of the theorem. Unfortunately so far the suggestions for such an elementary proof have been flawed. Current experiments agree with the spin-statistics theorem to high precision but despite the best efforts of many of the century's top physicists an understanding of the spin-statistics connection has remained elusive. One of the early proofs of the spin statistics theorem appeared in the PhD thesis of Dewey. He began his physical review article on his work with the summary "The problem of the connection between the spin and the statistics of particles was first tackled by Pauli. His work was not correct ...". The rest of this thesis will be an investigation of an elementary non-relativistic construction of spin-statistics. As such I hope that it contributes to the understanding of the spin-statistics theorem. The explanation it provides will not be complete although I believe that will still leave me in good company.

Chapter 2

Representation Theory

This chapter summarises the results in the representation theory of the symmetric and unitary groups that are used subsequently. Anyone already familiar with this material should skip this chapter, although the final section on the Littlewood-Richardson theorem is probably not widely known and will be used extensively later.

I will present the representation theory of $SU(n)$ and S_n in parallel, in order to emphasise the similarities between the two treatments that will lead to both being classified by Young tableau. The examples are chosen to fit with the later chapters and so may not appear as in a standard text.

2.1 Preliminaries

An *isomorphism* is a map between two sets which preserves the structure of the domain of the map. In the sets with which we will be concerned this structure will in general be a multiplication law on the set. An *automorphism* is an isomorphism from a set back into itself so the domain and range of the map are the same.

An *algebra* of order n over the complex numbers is defined by n^3 complex numbers γ_{ijk} where $1 \leq i,j,k \leq n$. Elements of the algebra are sets of n complex numbers $\mathbf{x} = [x_1, \ldots, x_n]$. Addition, multiplication and scalar multiplication are defined according to the following rules

$$a\mathbf{x} + b\mathbf{y} = [ax_1 + by_1, \dots ax_n + by_n] \tag{2.1}$$

$$\mathbf{x}.\mathbf{y} = [\sum \gamma_{ij1} x_i y_j, \dots, \sum \gamma_{ijn} x_i y_j] \tag{2.2}$$

a and b are complex numbers. An algebra is associative if $\mathbf{x}(\mathbf{yz}) = (\mathbf{xy})\mathbf{z}$. All the algebras we will meet are associative.

2.2 Groups and representations

A *group*, G, is a set of elements, g, closed under an associative multiplication law, containing an identity element and with each element having a unique inverse. So with I the identity element, $g^{-1}g = I$. The *order* of the group G is the number of elements in G and is denoted by Ω_G. If the multiplication law of a group is commutative as well as associative the group is called *Abelian*.

A *matrix representation* of the group, $T(G)$, is a map from G to a set of finite dimensional complex matrices which preserves the multiplication law of the group, so that

$$T(g_1)T(g_2) = T(g_1 g_2) \tag{2.3}$$

The dimension of the representation is the dimension of the vector space acted on by the matrices. The trivial representation maps all elements of the group to unity (regarded as a one dimensional matrix).

Given two representations of a group, $T(G)$ and $S(G)$, a third representation can be formed by taking the direct sum of these two representations.

$$T(g) \oplus S(g) = \begin{pmatrix} T(g) & 0 \\ 0 & S(g) \end{pmatrix} \tag{2.4}$$

T and S form blocks on the diagonal of the new representation and the rest of the entries are zero. Any representation that can be brought into this block diagonal form by a change of basis is called a *reducible* representation, and conversely those representations which can't be further decomposed into diagonal blocks are called the *irreducible* representations of the group. Starting with a reducible representation we can look for a transformation which divides the representation into two representations of lower dimension as in equation (2.4). The same process can then be repeated for the constituent representations. If they are reducible there exists a transformation which brings them into block diagonal form. As the matrix has finite dimension the procedure must end with the representation expressed as a direct sum of a set of irreducible representations.

An alternative intrinsic definition of irreducibility can be made by considering the space the representation acts on. We will take the representation to be n dimensional. Then if the exists a subspace of dimension m where $m < n$ which is invariant under all transformations of the group the representation is reducible. If there is no invariant subspace the representation is irreducible. For a deeper discussion of reducibility see [34] chapter 3.

A second way to construct a new representation of a group is to use the matrix tensor product. This is the *direct product* representation.

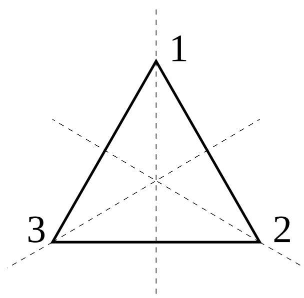
$$[(T \otimes S)(g)]_{ix,jy} = [T(g)]_{ij} \otimes [S(g)]_{xy} \tag{2.5}$$

The matrix representation can be constructed by taking $T(g)$ and replacing the term $T_{ij}(g)$ with the matrix $T_{ij}(g)S_g$. If the representations T and S are of n and m dimensions respectively the resulting matrices are of dimension $n \times m$. Clearly the multiplication law (2.3) is preserved for $(T \otimes S)$ so the tensor product defines a representation. The direct product representation is not in general irreducible even when the representations used in the product are irreducible.

A *subgroup* H of G is a subset of the elements of G obeying all the requirements of a group. So, for example, all subgroups contain the identity. A representation of G must also be a representation of H, as the multiplication law (2.3) holds for all $g \in H$. In this way we can consider restricting a irreducible representation of G to the elements of H. The representation of H defined in this way will in general be reducible. It is the decomposition into irreducible representations of H of a restriction of a representation of G that is the central problem of this thesis.

2.2.1 The symmetry group of an equilateral triangle

For our first example of a group we will consider the symmetries of an equilateral triangle.



Symmetry operations leave the triangle unchanged but permute the labels of the vertices. Reflecting in the three symmetry axes exchanges pairs of vertices. The diagram can also be rotated by $2\pi/3$ clockwise or anti-clockwise which permutes all three vertices. Leaving the diagram unchanged is the identity operation of this symmetry group. The group of symmetry operations has six elements which naturally fall into three classes, the identity, reflections and rotations.

2.2.2 Cosets and the quotient group

Given a group G with a subgroup H we can define sets of elements of G by taking an element $g \in G$ and multiplying all the elements of H by it. This set of elements of G is called a *left coset* of G and is denoted gH. Right cosets can also be defined by multiplying by g on the right.

For a finite group G divides into distinct cosets of H. Take an element g not in H, none of the elements of gH are in H as otherwise $gh_1 = h_2$ so $g = h_1^{-1}h_2$ and is in H contrary to the assumption. All the elements of gH are also different as $gh_1 = gh_2$ implies $h_1 = h_2$. We now have two distinct sets of Ω_H elements H

and gH in G. If the group G has not been exhausted we can continue by selecting some element g' not in H or gH. In this way G breaks down into a finite number of distinct cosets each of Ω_H elements

$$G = H + gH + g'H + \ldots \tag{2.6}$$

We see that the order of the subgroup Ω_H must divide the order of the group Ω_G.

An *invariant* subgroup of G is a group $H \subset G$ for which

$$gHg^{-1} = H \tag{2.7}$$

gHg^{-1} is the set of elements ghg^{-1} where h runs through the subgroup H. For an invariant subgroup H we know that $gH = Hg$ from which we can define a multiplication law for the cosets

$$(aH)(bH) = a(Hb)H = a(bH)H = ab(HH) = (ab)H \tag{2.8}$$

where we used the fact that H is a subgroup to deduce that $HH = H$. The inverse of a coset aH is $a^{-1}H$ so the cosets of an invariant subgroup form a group themselves. This group is called the *quotient group* and is denoted by G/H.

2.3 Classes and characters

For an element $g \in G$ the *class* of g is the set of elements $g' \in G$ that can be obtained from g by conjugating with another element $h \in G$, i.e.

$$g' = h^{-1}gh \tag{2.9}$$

If an element g' is conjugate to g and g'' is conjugate to g' then g and g'' are also conjugate. For example if we have $g' = h_1^{-1}gh_1$ then

$$g'' = h_2^{-1}h_1^{-1}gh_1h_2 = (h_1h_2)^{-1}g(h_1h_2) \tag{2.10}$$

The elements of a group can be partitioned into these disjoint conjugacy classes. The number of elements in each class is the order of the class. The identity element forms a separate class of order one for every group. It can be shown that the number of irreducible representations of a group is equal to the number of classes of the group.

Consider the example of the symmetries of the equilateral triangle. We will take h to be a reflection. For a reflection we know that h^{-1} is h. If we take g to be a clockwise rotation by $2\pi/3$ conjugating g by h we obtain $g' = h^{-1}gh$ which is an anti-clockwise rotation by $2\pi/3$. Both rotations are in the same conjugacy class. In fact we find that the three sets of symmetries, rotations, reflections and the identity form the conjugacy classes of this group. We will see later that there are also three irreducible representations of this symmetric group.

The classes of a group are important when we consider a representation of the group. For two elements in the same class

$$T(g') = T(h)^{-1}T(g)T(h) \tag{2.11}$$

Then taking the trace of both sides by summing the diagonal matrix elements we find that

$$Tr\, T(g') = Tr\, T(g) \tag{2.12}$$

The trace of a representation is a function of the classes of the group and is called the *character* of the representation,

$$\chi_T(g) = Tr\, T(g) \tag{2.13}$$

We have already seen that a representation can be decomposed in to a direct sum of irreducible representations

$$T(g) = \bigoplus T_j(g) = \begin{pmatrix} T_1(g) & & & \\ & T_2(g) & & \\ & & \ddots & \\ & & & T_m(g) \end{pmatrix} \tag{2.14}$$

where T_j is an irreducible representation of G. Taking the trace of T we see that

$$\chi_T(g) = \sum_j \chi_{T_j}(g) \tag{2.15}$$

Any character can be written as a sum of the characters of the irreducible representations of the group.

The irreducible characters have some important properties which will be used subsequently. A proof of these relations can be found in chapter 3 of [34] or any other introduction to group theory. Firstly the irreducible characters are orthogonal if averaged over the group,

$$\frac{1}{\Omega_G}\sum_g \overline{\chi}_j(g)\chi_k(g) = \delta_{jk} \tag{2.16}$$

j and k label irreducible representations of G. $\overline{\chi}(g)$ is the complex conjugate of the character. As the character is a function of the classes of G if we label the classes of G by λ and take g_λ to be an element of λ we can rewrite (2.16) as

$$\frac{1}{\Omega_G}\sum_\lambda \Omega_\lambda \overline{\chi}_j(g_\lambda)\chi_k(g_\lambda) = \delta_{jk} \tag{2.17}$$

where Ω_λ is the order of the class λ. If we consider the character X of a representation $T(g) = \bigoplus T_j(g)$, where the T_j are the irreducible representations of G, then from (2.16) N_k the multiplicity of the irreducible representation T_k in the decomposition of T is

$$N_k = \frac{1}{\Omega_G}\sum_g \overline{X}(g)\chi_k(g) \tag{2.18}$$

The orthogonality of irreducible characters can be used to decompose a general representation into its irreducible components.

The irreducible characters are also orthogonal when the characters of two classes are averaged over the irreducible representations.

$$\begin{aligned} \Omega_\lambda \sum_i \chi_i(\lambda)\overline{\chi}_i(\lambda) &= \Omega \\ \sum_i \chi_i(\lambda)\overline{\chi}_i(\rho) &= 0 \quad (\lambda \neq \rho) \end{aligned} \tag{2.19}$$

If the irreducible characters are recorded in a character table the two orthogonality relations (2.17) and (2.19) refer to the orthogonality of the rows and columns of the table respectively.

2.4 The symmetric group

The set of n distinct symbols $(\alpha_1, \ldots, \alpha_n)$ can be arranged in $n!$ orderings. A *permutation* ρ acts on the symbols by rearranging them into a new order,

$$\rho(\alpha_1, \ldots, \alpha_n) = (\alpha_{\rho(1)}, \ldots, \alpha_{\rho(n)}) \tag{2.20}$$

There is an identity permutation leaving the symbols in their original positions and all permutations have an inverse which undoes the change in order. Applying two permutations to the same set of symbols is equivalent to a single permutation of the symbols so the set of permutations is closed under multiplication and forms a group. This group of all possible permutations of the n symbols is called the symmetric group S_n. S_n is of order $n!$. The symmetries of the equilateral triangle correspond to the symmetric group S_3 where the symbols that are permuted in this case are the vertex labels.

A *cycle* $(ijk\ldots l)$ is a permutation in which the i'th symbol is moved to the j'th place the j'th to the k'th and so on. The l'th symbol replaces the i'th. The order of the cycle is the number of terms in the cycle. Every permutation can be written as a product of disjoint cycles. For example the permutation

$$(a, b, c, d, e, f) \rightarrow (d, f, a, c, e, b)$$

is the result of applying the cycles $(134)(26)$, cycles of order one are omitted by convention. A cycle of length m can be further factored into a product of $(m-1)$ two cycles. Consequently any permutation can be written as a product of these transpositions. A permutation, ρ, is then referred to as even if it can be written as the product of an even number of transpositions and odd if the number of transpositions required is odd. The product of two even or two odd permutations is even while the product of an even and an odd permutation is odd. The *sign* of a permutation $\mathrm{sgn}(\rho)$ is one for odd permutations and zero for even permutations.

For the symmetric groups two permutations which, when written as a product of disjoint cycles, have the same number of cycles of the same order are in the same conjugacy class. We can see this cycle structure of classes by considering conjugating a cycle ρ by σ, as in (2.9). For example if ρ is the cycle (123) then

$$\rho' = \sigma(123)\sigma^{-1} = (\sigma(1)\sigma(2)\sigma(3)) \tag{2.21}$$

The cycle ρ is effectively applied to a reordered set of symbols so for the conjugate element the length of the cycles is preserved. In S_3 the symmetry group of the equilateral triangle the three reflections written as cycles are (12), (13) and (23), the rotations are (123) and (132).

2.4.1 The irreducible representations of S_3

Continuing our example we will find the irreducible representations of S_3. The trivial representation, where all the elements are represented by 1, is an irreducible representation of S_3. The alternating representation of the symmetric group is defined by associating 1 to the even permutations, in this case the identity and three cycles, and -1 to the odd permutations, the two cycles. We can see that the alternating representation obeys the same multiplication law as the group, a reflection followed by a rotation is a reflection, two reflections is a rotation or the identity. This representation is one dimensional and so irreducible.

There is also a two dimensional irreducible representation of S_3. To find it we use the defining representation of the permutation group. The *defining* representation of a permutation ρ in S_n is the matrix $D(\rho)$ where

$$D_{ij}(\rho) = \begin{cases} 1 & \text{if} \quad \rho(i) = j \\ 0 & \text{otherwise} \end{cases} \tag{2.22}$$

For S_3 the defining representation is

$$D(I) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad D(12) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$D(13) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad D(23) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{2.23}$$

$$D(123) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad D(132) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

This representation is reducible, multiplying a vector (x, x, x) by any of these permutation matrices will leave the vector unchanged. It forms a one dimensional subspace which is invariant under the group transformations. The subspace transforms according to the trivial representation of S_3. We can bring this representation into block diagonal form by changing the basis. One possible choice of transformation is

$$P = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & -2 \end{pmatrix}$$

Then $P^{-1}DP$ decomposes into the trivial representation and the irreducible two dimensional representation, T.

$$T(I) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad T(12) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$T(13) = \begin{pmatrix} 1/2 & -3/2 \\ -1/2 & -1/2 \end{pmatrix} \quad T(23) = \begin{pmatrix} 1/2 & 3/2 \\ 1/2 & -1/2 \end{pmatrix} \tag{2.24}$$

$$T(123) = \begin{pmatrix} -1/2 & -3/2 \\ 1/2 & -1/2 \end{pmatrix} \quad T(132) = \begin{pmatrix} -1/2 & 3/2 \\ -1/2 & -1/2 \end{pmatrix}$$

We will see later that the irreducible representations of S_n are in one-to-one correspondence with the partitions of n and so these three representations are the only irreducible representations of S_3.

2.5 The group $SU(n)$

A unitary matrix is a matrix whose inverse is its Hermitian conjugate. $SU(n)$ is the group of $n \times n$ unitary matrices with determinant one. The group $SU(n)$ is a connected, compact Lie group acting on vectors in $\mathbb{C}^n$. Elements of the group can be parameterised by real numbers γ_j where

$$g = \exp(i \sum_j \gamma_j \Gamma_j) \tag{2.25}$$

The Γ_j are $n \times n$ hermitian traceless matrices and are called the *generators* of $SU(n)$. There are $n^2 - 1$ linearly independent traceless hermitian matrices so the group elements require $n^2 - 1$ coefficients γ_j to parameterise the elements.

The generators of the group are infinitesimal group elements,

$$\exp(i\epsilon\Gamma) \to I + i\epsilon\Gamma$$

for small ϵ. These infinitesimal group generators form a vector space and so are often easier to work with than the group elements themselves. If we look at the product of group elements

$$\exp(i\epsilon\Gamma_b)\exp(i\epsilon\Gamma_a)\exp(-i\epsilon\Gamma_b)\exp(-i\epsilon\Gamma_a) = I + \epsilon^2[\Gamma_a, \Gamma_b] + \dots$$

This product is also a group element and so can be written as $\exp(i\sum \gamma_c\Gamma_c)$. As $\epsilon \to 0$ we must have

$$[\Gamma_a, \Gamma_b] = i\, f_{abc}\Gamma_c \tag{2.26}$$

The constants f_{abc} are the *structure constants* of the group. The structure constants determine the multiplication law of the group. They obey the Jacobi identity

$$f_{bcd}f_{ade} + f_{abd}f_{cde} + f_{cad}f_{bde} = 0 \tag{2.27}$$

The structure constants also determine a representation of the group, the *adjoint representation*, $(\Gamma_a)_{ij} \equiv -if_{aij}$, see [27] chapter 2. The set of generators with commutation relations defines an algebra associated to the group, the Lie algebra $su(n)$.

2.5.1 SU(2)

A simple example of the structure constants for three generators is $f_{abc} = \varepsilon_{abc}$ the completely antisymmetric tensor. The commutation relations are then the angular momentum commutation relations. The *defining representation* of the Lie algebra is the $n \times n$ representation of the algebra which generates the group itself. For the angular momentum commutation relations traceless hermitian generators can be defined from the Pauli matrices

$$\sigma_x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.28}$$

These generate the group $SU(2)$.

2.5.2 Roots and weights of Lie algebras

A representation of the group defines a representation of the algebra and vice versa. This correspondence is natural but to actually show that representations are connected in this way takes some work, a good discussion is found in chapter 4 of [40]. We will turn now to discuss an irreducible representation $T(SU(n))$ associated with a representation $T(su(n))$ of the algebra.

From the set of $n^2 - 1$ generators we select a *Cartan sub-algebra*. This is a maximal set of $n-1$ commuting generators, which we will label H_i. The space the representation $T(su(n))$ acts on will be spanned by eigenvectors of the Cartan sub-algebra and the eigenvalues with respect to the Cartan sub-algebra will be used to label these eigenvectors.

$$H_i \left|\boldsymbol{\mu}, T\right\rangle = \mu_i \left|\boldsymbol{\mu}, T\right\rangle \tag{2.29}$$

The eigenvalues μ_i are the *weights* of a representation and the vector $\boldsymbol{\mu}$ with components μ_i is a *weight vector*. The order of the terms in $\boldsymbol{\mu}$ is arbitrary, the results apply equally to any ordering.

The adjoint representation is the representation obtained by taking the generators as the basis of the space the algebra acts on. A generator then acts on a basis vector by commutation

$$\Gamma_i \, |\Gamma_j\rangle = |[\Gamma_i, \Gamma_j]\rangle \tag{2.30}$$

The Cartan sub-algebra corresponds to a set of vectors with zero weight

$$H_i \, |H_j\rangle = 0 \tag{2.31}$$

Diagonalising the space acted on by the generators gives a set of states $|Y_{\boldsymbol{\alpha}}\rangle$ where

$$H_i \, |Y_{\boldsymbol{\alpha}}\rangle = \alpha_i |Y_{\boldsymbol{\alpha}}\rangle \tag{2.32}$$

The states correspond to generators $Y_{\boldsymbol{\alpha}}$ so that

$$[H_i, Y_{\boldsymbol{\alpha}}] = \alpha_i Y_{\boldsymbol{\alpha}} \tag{2.33}$$

The Y_α are linear combinations of the generators not in the Cartan sub-algebra. These weight vectors $\boldsymbol{\alpha}$ of the adjoint representation with components α_i are the *roots* of the Lie algebra.

Let us see how an operator $Y_{\boldsymbol{\alpha}}$ acts on a vector in a general representation $T(su(n))$.

$$\begin{aligned} H_i Y_{\boldsymbol{\alpha}} \, |\boldsymbol{\mu}, T\rangle &= [H_i, Y_{\boldsymbol{\alpha}}] \, |\boldsymbol{\mu}, T\rangle + Y_{\boldsymbol{\alpha}} H_i \, |\boldsymbol{\mu}, T\rangle \\ &= (\boldsymbol{\mu} + \boldsymbol{\alpha})_i \, Y_{\boldsymbol{\alpha}} \, |\boldsymbol{\mu}, T\rangle \end{aligned} \tag{2.34}$$

We see that the vector $Y_{\boldsymbol{\alpha}}|\boldsymbol{\mu}, T\rangle$ is labelled by the weight vector $\boldsymbol{\mu} + \boldsymbol{\alpha}$. We have now identified the root vectors with raising and lowering operators for the weights of a Lie algebra. By choosing an initial state and repeatedly applying raising or lowering operators we find that

$$\frac{\boldsymbol{\alpha}.\boldsymbol{\mu}}{\boldsymbol{\alpha}^2} = -\frac{1}{2}(p - q) \tag{2.35}$$

p and q are the number of times the operator $Y_{\boldsymbol{\alpha}}$ or $Y_{-\boldsymbol{\alpha}}$ can be applied to $\boldsymbol{\mu}$ before reaching zero. Both p and q depend on the choice of weight $\boldsymbol{\mu}$ and root $\boldsymbol{\alpha}$. Equation (2.35) is derived in chapter 6 of [27], it is equivalent to the condition that angular momentum eigenvalues be integer or half integer.

2.5.3 The highest weight classification of irreducible representations

To provide a definition of a positive weight we fix the order of the generators in the Cartan sub-algebra. This fixes the order of the components of the weight vector. We then define a weight vector to be positive if its first non zero term is positive. With this definition we can order the weight vectors, $\boldsymbol{\mu}_1 > \boldsymbol{\mu}_2$ if $\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2 > 0$. The *highest weight vector* of a representation is greater than other weight vectors and is non-degenerate.

To provide a nontrivial example we introduce generators of the defining representation of $SU(4)$

$$
\begin{aligned}
S_{1i} &= \begin{pmatrix} \sigma_i & 0 \\ 0 & 0 \end{pmatrix} \qquad S_{2i} = \begin{pmatrix} 0 & 0 \\ 0 & \sigma_i \end{pmatrix} \\
E_x &= \frac{1}{2\sqrt{2}} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \qquad E_y = \frac{1}{2\sqrt{2}} \begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix} \\
E_z &= \frac{1}{2\sqrt{2}} \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}
\end{aligned}
\tag{2.36}
$$

σ_i are the Pauli matrices defined in (2.28) and I is the 2×2 identity matrix. To provide all the 15 generators of $SU(4)$ we must also include the commutators of these matrices. We can select the Cartan sub-algebra to be the set of diagonal matrices E_z, S_{1z} and S_{2z} in that order. A weight vector then consists of the eigenvalues

(e_z, s_{1z}, s_{2z}). A basis for the space the representation acts on is

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{x}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{x}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \mathbf{x}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{2.37}$$

As the Cartan sub-algebra is diagonal these basis vectors are eigenvectors of the sub-algebra and can be labelled by their weights.

$$\begin{aligned} \boldsymbol{\mu}_1 &= \left(\tfrac{1}{2\sqrt{2}}, \tfrac{1}{2}, 0\right) & \boldsymbol{\mu}_2 &= \left(\tfrac{1}{2\sqrt{2}}, -\tfrac{1}{2}, 0\right) \\ \boldsymbol{\mu}_3 &= \left(-\tfrac{1}{2\sqrt{2}}, 0, \tfrac{1}{2}\right) & \boldsymbol{\mu}_4 &= \left(-\tfrac{1}{2\sqrt{2}}, 0, -\tfrac{1}{2}\right) \end{aligned} \tag{2.38}$$

Using the definition of a positive weight vector

$$\boldsymbol{\mu}_1 > \boldsymbol{\mu}_2 > \boldsymbol{\mu}_3 > \boldsymbol{\mu}_4$$

$\boldsymbol{\mu}_1$ is the highest weight vector of the defining representation of $SU(4)$.

By defining which weights are positive we are also provided with a classification of raising and lowering operators depending on whether the associated root vector is positive or negative. A *simple root* is a positive root which can not be written as the sum of two positive roots. A positive root which is not simple can be written as the sum of two positive roots and either these are simple or we can repeat the procedure. Using this scheme we see that any positive root can be written as a sum of simple roots with positive integer coefficients. From condition (2.35) it can be shown that the simple roots are linearly independent and span the space of weight vectors, [27] chapter 8. Therefore there are the same number of simple roots as generators in the Cartan sub-algebra. Starting from the simple roots all other roots can be found by combining the roots and checking the condition (2.35).

By definition operating on the highest weight of a representation with a raising operator labelled by a positive root must give zero. As every positive root is a sum of simple roots it is sufficient to consider only the simple roots. We will label the simple roots $\boldsymbol{\alpha}^j$, $j = 1 \ldots m$. Substituting $p = 0$ for a heighest weight into (2.35) we have

$$\frac{2\boldsymbol{\alpha}^j.\boldsymbol{\mu}}{(\boldsymbol{\alpha}^j)^2} = q^j \tag{2.39}$$

The q^j's are non-negative integers. If a highest weight $\boldsymbol{\mu}^k$ is the weight of a representation where $q^k = 1$ and $q^j = 0$ for $j \neq k$ then $\boldsymbol{\mu}^k$ is called a *fundamental weight.* Any highest weight can be written as a sum of these fundamental weight vectors

$$\boldsymbol{\mu} = \sum_k q^k \boldsymbol{\mu}^k \tag{2.40}$$

Each highest weight labels an irreducible representation of $SU(n)$. The representations with highest weights $\boldsymbol{\mu}^k$ are called the fundamental representations. It can be shown that the irreducible representations of $SU(n)$ can be constructed from tensor products of the fundamental representations where the integers q^k are the multiplicity of the fundamental representations in the product.

The roots are the differences between the weights and the simple roots are the minimum positive differences. From the weights of the defining representation of $SU(4)$ (2.38) the simple roots of the group are

$$\begin{aligned} \boldsymbol{\alpha}^1 &= \boldsymbol{\mu}_1 - \boldsymbol{\mu}_2 = \left(0, 1, 0\right) \\ \boldsymbol{\alpha}^2 &= \boldsymbol{\mu}_2 - \boldsymbol{\mu}_3 = \left(\tfrac{1}{\sqrt{2}}, -\tfrac{1}{2}, -\tfrac{1}{2}\right) \\ \boldsymbol{\alpha}^3 &= \boldsymbol{\mu}_3 - \boldsymbol{\mu}_4 = \left(0, 0, 1\right) \end{aligned} \tag{2.41}$$

We can see that the roots $\boldsymbol{\alpha}^1$ and $\boldsymbol{\alpha}^3$ raise the S_{1z} and S_{2z} eigenvalues by one respectively and so correspond to the operators S_{1+} and S_{2+}, constructed from the generators in the usual way. The raising operator $Y_{\boldsymbol{\alpha}^2}$ is $[E_+, S_{2-}]$. With the simple roots we can use (2.39) to find the fundamental weights.

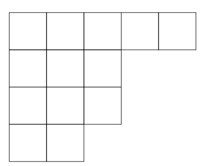
$$\begin{aligned} \boldsymbol{\mu}^1 &= \left(\tfrac{1}{2\sqrt{2}}, \tfrac{1}{2}, 0\right) \\ \boldsymbol{\mu}^2 &= \left(\tfrac{1}{\sqrt{2}}, 0, 0\right) \\ \boldsymbol{\mu}^3 &= \left(\tfrac{1}{2\sqrt{2}}, 0, \tfrac{1}{2}\right) \end{aligned} \tag{2.42}$$

The exact form of the simple roots and fundamental weights depends on the definition of the generators and the choice of the Cartan sub-algebra. Defining the simple roots of $SU(4)$ in this form will simplify the work later.

2.6 Representations of the symmetric group and Young tableau

2.6.1 Partitions, graphs and tableau

We turn now to finding the irreducible representations of the symmetric group. The construction uses tableau and it is these graphs of partitions which we will introduce first. Let $\boldsymbol{\lambda}$ be a *partition* of an integer n into m integer parts, $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ where $\lambda_1 + \lambda_2 + \cdots + \lambda_m = |\boldsymbol{\lambda}| = n$ and the order of terms is immaterial. If the number of one's in $\boldsymbol{\lambda}$ is a, the number of two's b etc then the partition can be written $\boldsymbol{\lambda} = (1^a, 2^b, 3^c, \ldots)$. We have already seen that the class of an element of the symmetric group is determined by the number of cycles of each order and so the classes of S_n are labelled by partitions of n. As partitions don't depend on the order of the λ_i we can adopt the convention that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$. Then to each partition we associate a graph or tableau with λ_1 boxes in the first row, λ_2 boxes in the second and so on, the rows being aligned on the left. So for example the partition $(5, 3, 3, 2)$ of 13 is associated with the graph



We will keep to the convention that a graph refers only to an empty sequence of boxes. When the boxes are labelled by symbols the figure will be referred to as a tableau. In some cases the symbols that distinguish the boxes may be suppressed when the figure is drawn although in the text it will still be referred to as a tableau. We will see that not only are the classes of S_n labelled by graphs of n boxes but so are the irreducible representations of S_n.

2.6.2 Characteristic units of the group algebra

The *Frobenius algebra* of the symmetric group is the algebra obtained by taking the elements of the group as the basis of the algebra. Then the multiplication law of the group determines a product of the basis elements. If ρ_j is an element of the group an element τ of the algebra can be written

$$\tau = \sum_j \xi_j \rho_j \tag{2.43}$$

where the ξ_j are complex coefficients. The group algebra has many interesting properties, see [40] chapter 4. We will state a few of relevance here without proof.

The group algebra is isomorphic to a direct sum of matrix sub-algebras, subsets of the elements of the algebra closed under multiplication. Each of these sub-algebras defines a representation of the group as a group element can be expressed as a sum of elements of the sub-algebra. The number of matrix sub-algebras in the group algebra is equal to the number of classes of the group and the representations of S_n defined by the sub-algebras are the irreducible representations. The group algebra is an algebra that contains all the irreducible representations of the group when written as a sum of matrix sub-algebras. Each irreducible representation appears with multiplicity equal to the dimension of the representation.

There are particular elements of the group algebra that are associated with the representations of the group. These *characteristic units* of the algebra are the idempotent elements τ such that $\tau^2 = \tau$. We can express τ as a sum of elements of the irreducible matrix sub-algebras

$$\tau = \sum \tau_j \tag{2.44}$$

where j labels the sub-algebra. As τ_j is idempotent it can be transformed into a diagonal matrix $(1^{r_i}, 0^{l-r_i})$. Multiplying a group element by this characteristic unit and taking the trace we obtain a compound character of the group.

$$X = \sum r_j \chi_j \tag{2.45}$$

A *primitive characteristic unit* is the characteristic unit associated with the irreducible character χ_k, $r_k = 1$ and $r_j = 0$ for $j \neq k$. From the definition of the primitive characteristic units we see that any characteristic unit can be written as a sum of primitive characteristic units. Two primitive characteristic units of the same matrix sub-algebra are transforms of each other. Finally we state a lemma used later

Lemma 2.6.1. *The product of two primitive characteristic units is either zero, nilpotent, or a multiple of a primitive characteristic unit.*

If the two primitive characteristic units are in different sub-algebras the product is zero. If they are in the same sub-algebra and the product is not zero then, as the primitive characteristic units both have rank one, the product must also have rank one (the rank of the matrix is the number of linearly independent rows or columns). The reduced characteristic equation of the product is a quadratic with a zero root, $x^2 - \lambda x = 0$. If $\lambda = 0$ the product is nilpotent ($x^2 = 0$), otherwise it is a multiple of a primitive characteristic unit.

We have seen that the irreducible characters of the group are associated with primitive characteristic units which are linear combinations of the group elements. We will show that the characteristic units of S_n can be constructed by using tableau and that they are also used to construct irreducible representations of the special unitary groups.

2.6.3 Characteristic units of S_n.

The symmetrisation operator acting on r symbols is the sum of the group elements of the symmetric group that permute the symbols. It is an element of the group algebra of a symmetric group acting on the symbols. The antisymmetrisation operator sums the same group elements but with a minus sign attached to odd permutations.

We take the graph of the partition $\boldsymbol{\lambda}$ of n and assign to every box one of the integers from 1 to n this is a Young tableau introduced by Young in [54]. The integers are just one choice of a set of n symbols and so the order that the integers are assigned to boxes is not significant.

Let P_i be the symmetrisation operator for the λ_i symbols in the i'th row of the tableau. Then P is defined to be the product of the m row symmetrisation operators. Multiplying P_i by a permutation in P_i produces the same symmetrisation operator P_i. So $P_i/\lambda_i!$ is an idempotent element of the group algebra. Consequently $P/\lambda_1!\ldots\lambda_m!$ is also idempotent and therefore also a characteristic unit. Similarly we define N as the product of the antisymmetrisation operators of the symbols in each column. $N/\overline{\lambda}_1!\ldots\overline{\lambda}_q!$ is also a characteristic unit of S_n, $\overline{\lambda}_i$ is the length of the i'th column of $\boldsymbol{\lambda}$.

Neither of these characteristic units is primitive but if both are written as a sum of primitive characteristic units they have only a single primitive characteristic unit in common, see [40] chapter 5. The primitive unit in both N and P is associated with the character $\chi^{\boldsymbol{\lambda}}$. From lemma (2.6.1) the product of two primitive characteristic units is zero, nilpotent or a multiple of a primitive characteristic unit. The product $NP/\overline{\lambda}_1!\ldots\overline{\lambda}_q!\lambda_1!\ldots\lambda_m!$ contains the identity element of S_n. It can not therefore be zero or nilpotent and so is a multiple of the primitive characteristic unit $\chi^{\boldsymbol{\lambda}}$. The normalised primitive characteristic unit is

$$\frac{\chi^{\boldsymbol{\lambda}}(I)}{\Omega_{S_n}}NP$$

$\chi^{\boldsymbol{\lambda}}(I)$ is the dimension of the irreducible representation $\boldsymbol{\lambda}$. We have associated the irreducible representations of S_n to the partitions $\boldsymbol{\lambda}$ of n via the primitive characteristic units generated by a Young tableau of shape $\boldsymbol{\lambda}$.

As primitive characteristic units are a central feature of the later chapters it is worth seeing an example for S_3. An irreducible representation labelled by the partition $(2,1)$ is associated with a tableau

1	2
3	

Any other labelling of the boxes is equally valid. We can now write down P and N for the tableau,

$$P = I + (12) \tag{2.46}$$

$$N = I - (13) \tag{2.47}$$

$$\frac{\chi^{(21)}(I)}{6}NP = \frac{1}{3}(I + (12) - (13) - (123)) \tag{2.48}$$

It can be verified that this is an idempotent element of the group algebra. By construction it is associated with an irreducible representation labelled by the partition $(2,1)$.

2.6.4 Constructing representations of $SU(n)$ with characteristic units

We have seen that Young tableau of n boxes label the irreducible representations of S_n. The characteristic units they generate can also be used to construct irreducible representations of the special unitary groups. We will construct representations of $SU(n)$ from tensor products of the defining representation, this is the n dimensional irreducible representation. The defining representation will have n weight vectors for its basis vectors $\boldsymbol{\nu}^1 \ldots \boldsymbol{\nu}^n$ ordered so that $\boldsymbol{\nu}^1 > \boldsymbol{\nu}^2 > \cdots > \boldsymbol{\nu}^n$. The differences between adjacent weights provide the simple roots of $SU(n)$,

$$\boldsymbol{\alpha}^i = \boldsymbol{\nu}^i - \boldsymbol{\nu}^{i+1} \tag{2.49}$$

The generators can be normalised so the weights all have the same length and the angle between weight vectors is the same for any pair. For the roots

$$\boldsymbol{\alpha}_i.\boldsymbol{\alpha}_j = \begin{cases} 1 & i = j \\ -\frac{1}{2} & j = i+1 \text{ or } i-1 \\ 0 & i \neq j \text{ or } j \pm 1 \end{cases} \tag{2.50}$$

This can be verified for the roots of the $SU(4)$ defined in (2.42).

The fundamental weights are given by a sum of the weights of the defining representation,

$$\boldsymbol{\mu}^j = \sum_{k=1}^{j} \boldsymbol{\nu}^k = j\,\boldsymbol{\nu}^1 + \sum_{k=1}^{j-1} k\,\boldsymbol{\alpha}^k \tag{2.51}$$

From the condition (2.39) they must obey the relation

$$\frac{2\boldsymbol{\alpha}^i.\boldsymbol{\mu}^j}{(\boldsymbol{\alpha}^i)^2} = \delta_{ij} \tag{2.52}$$

and as the defining representation is a fundamental representation we can use the results for multiplying roots (2.50) to check (2.51).

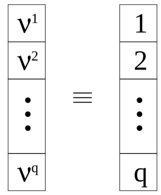
If we consider constructing a new representation by taking the tensor product of q defining representations it is clear that the the highest weight vector would be the tensor product of q identical vectors all labelled by $\boldsymbol{\mu}^1$. The representation is not irreducible but it must contain the irreducible representation with highest weight $q\boldsymbol{\mu}^1$. If the tensor product is symmetrised then the tensor product of q $\boldsymbol{\mu}^1$'s will still have the same highest weight vector although the dimension of the representation has been reduced. The symmetrised tensor product is an irreducible representation of $SU(n)$, as it is the representation of least dimension which has highest weight $q\boldsymbol{\mu}^1$. Symmetrising a tensor product of q terms is equivalent to acting on the tensor product with the primitive characteristic unit of S_q labelled by a tableau with a single row.

Rather than symmetrising we could consider antisymmetrising a tensor product of q defining representations. Permutations of a tensor product of basis vectors of the defining representation are added with the vectors resulting from odd permutations acquiring a minus sign. This is also the action of a primitive characteristic unit of S_q labelled by a tableau of a single column. In this case it is clear that a tensor product of q identical basis vectors $\boldsymbol{\mu}^1$ will be zero as each odd permutation will cancel an even one. To be non zero the highest weight vector must be a product of q different basis vectors of the defining representation, the vectors $\boldsymbol{\nu}^1, \ldots, \boldsymbol{\nu}^q$. The highest weight vector will have weight

$$\boldsymbol{\mu} = \sum_{i=1 \ldots q} \boldsymbol{\nu}^i, \tag{2.53}$$

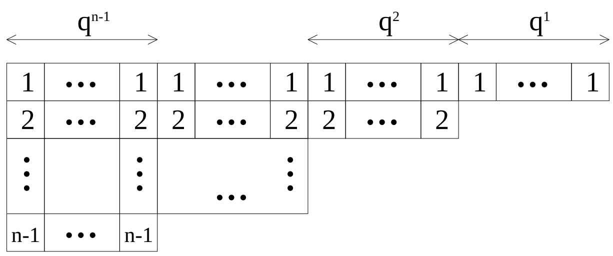
which is the highest weight of the fundamental representation, $\boldsymbol{\mu}^q$, see equation (2.51).

The fundamental representations of $SU(n)$ are recorded using a Young tableau for the characteristic unit of S_q used to construct the highest weight of the representation.



Rather than fill in the tableau with basis vectors $\boldsymbol{\nu}^i$ just the labels i will be used to denote the vectors as in the example above.

We have already noted that a general highest weight can be constructed from the fundamental weights, equation (2.40). We want to construct an arbitrary highest weight vector of an irreducible representation from the defining representation. It should have a highest weight $\boldsymbol{\mu} = \sum_i q^i \boldsymbol{\mu}^i$. This highest weight vector will be recorded in the tableau



(2.54)

There are $n-1$ fundamental weights that the representation can be constructed from, and each is repeated q^i times in the tableau so this has the correct weight.

The tableau 2.54 is of a partition $\boldsymbol{\lambda}$ where $\lambda_k = \sum_{j=k}^{n-1} q^j$. To construct the representation of $SU(n)$ the primitive characteristic unit of the symmetric group associated with the tableau $\boldsymbol{\lambda}$ is constructed. If there are q boxes in the tableau this is a characteristic unit of S_q. The characteristic unit $\chi^{\boldsymbol{\lambda}}(I)NP/\Omega_{S_q}$ consists of a set of symmetry conditions which can then be applied to the tensor product of defining representations. The symbols $1 \ldots q$ used to construct the characteristic unit correspond to the terms in the tensor product of the q basis vectors of the defining representation of $SU(n)$. A basis vector of the irreducible representation $\boldsymbol{\lambda}$ of $SU(n)$ is constructed by assigning basis vectors of the defining representation to boxes of the tableau. The product is symmetrised with respect to the terms in the same row then antisymmetrised with respect to those in the same column. The order of terms in the tensor product has not been specified but neither has the labelling of the boxes in the tableau for the primitive characteristic unit so this ambiguity is not important.

To see that (2.54) has the highest weight possible for the given shape of tableau we can consider replacing one of the vectors $\boldsymbol{\nu}^i$ with a vector with higher weight, $\boldsymbol{\nu}^j$ where $j < i$. As each column contains all the $\boldsymbol{\nu}^j$ with $j < i$ antisymmetrising the column gives zero, no state with higher weight can be constructed using that shape of tableau.

This procedure enables us to construct irreducible representations of $SU(n)$ from any Young tableau of up to $n-1$ rows. Representations with all the possible highest weights can be constructed this way so we can construct all the irreducible representations of $SU(n)$. If we attempted to construct an irreducible representation of $SU(n)$ using a tableau with more than n rows each column would need to be filled with different basis vectors of the defining representation. The defining representation has only n basis vectors so this is not possible and no representations can be constructed this way. A tableau with n rows can be used to construct an irreducible representation of $SU(n)$. Each of the columns of length n will contain all n of the basis vectors of the defining representation. These terms will be present in any vector of the representation and so this is the same representation as that which is constructed using the tableau with the columns of length n removed. The irreducible representations of $SU(n)$ are labelled by and can be constructed from Young tableau with up to $n-1$ rows. Columns of n boxes can be added to the tableau without changing the representation of $SU(n)$.

2.7 Characters of $U(n)$ and $SU(n)$.

If we return to thinking of the tableau as a partition then we label the irreducible representations of $SU(n)$ with a string of integers $\mathbf{f} = (f_1, \ldots, f_{n-1})$. $|\mathbf{f}| = f_1 + f_2 + \cdots + f_{n-1}$ is the total number of boxes in the tableau. $\mathbf{f}$ is one of the possible partitions of $|\mathbf{f}|$ though not all partitions label irreducible representations of $SU(n)$ as some partitions could be into more than n parts. The characters of $U(n)$ and

$SU(n)$ are functions of the classes of the groups and these functions will depend on the irreducible representation, labelled by $\mathbf{f}$. To define these functions we first parameterise the classes of the group. We will deal with $U(n)$ and specialise the results to $SU(n)$ where necessary. A full account is found in [53] or [29].

An element $Q \in U(n)$ is conjugate to a diagonal element where the diagonal terms have modulus one, see [53] chapter 7. So

$$U^{-1}QU = \begin{pmatrix} \epsilon_1 & & & \\ & \epsilon_2 & & \\ & & \ddots & \\ & & & \epsilon_n \end{pmatrix} \tag{2.55}$$

for some unitary matrix U and $|\epsilon_i| = 1$ for all i. Let $\epsilon_i = e^{i\phi_i}$ then the n angles ϕ parameterise the classes of $U(n)$. We will use $\boldsymbol{\epsilon}$ to refer to the set of diagonal elements $(\epsilon_1, \epsilon_2, \ldots, \epsilon_n)$. The class is invariant under a permutation of the diagonal terms so $\boldsymbol{\epsilon}$ is unordered.

As $U(n)$ is a continuous group to sum a function of the classes over the group we associate a volume element to the parameters $\boldsymbol{\epsilon}$ corresponding to the proportion of the space of unitary matrices in the class. This is the uniform Haar measure on the unitary group. Let

$$\Delta = \prod_{i<k} (\epsilon_i - \epsilon_k) \tag{2.56}$$

then the proportion of the group parameterised by angles lying between ϕ_i and $\phi_i + d\phi_i$ is $\Delta\overline{\Delta}\, d\phi_1 \ldots d\phi_n$. A class function of the unitary group is a symmetric function of the angles and with this definition of a measure on the group we can define the average of a function $g(\boldsymbol{\epsilon})$.

Theorem 2.7.1. *The average of a class function $g(\phi_1, \ldots, \phi_n)$ is*

$$\frac{1}{\Omega} \int_0^{2\pi} \cdots \int_0^{2\pi} g\, \Delta\overline{\Delta}\, d\phi_1 \ldots d\phi_n$$

where

$$\Omega = \int_0^{2\pi} \cdots \int_0^{2\pi} \Delta\overline{\Delta}\, d\phi_1 \ldots d\phi_n$$

The class functions that we will average over the group are the characters of $U(n)$ and its subgroups. The character is the trace of the matrix representation $T(U(n))$. As it is only a function of the class of the group elements not the elements themselves we only need evaluate it on the diagonal elements of the group $\{\boldsymbol{\epsilon}\}$. The subgroup of diagonal elements is Abelian (the multiplication law is commutative),

$$T(\phi_1,\ldots,\phi_n)T(\phi'_1,\ldots,\phi'_n) = T(\phi_1+\phi'_1,\ldots,\phi_n+\phi'_n) \tag{2.57}$$

The representation decomposes into a sum of one dimensional representations $Z_{\mathbf{k}}(\boldsymbol{\epsilon})$ where $|Z_{\mathbf{k}}(\boldsymbol{\epsilon})| = 1$, as in chapter 7 of [53]. These representations obey the multiplication law (2.57). Solving for one dimensional functions of $\boldsymbol{\epsilon}$

$$Z_{\mathbf{k}} = \epsilon_1^{k_1}\ldots\epsilon_n^{k_n} \tag{2.58}$$

The characters of representations of $U(n)$ are sums of monomials with integer coefficients.

The characters are symmetric functions of the angles. They can however be associated with antisymmetric functions $\chi\Delta$, which appear naturally when averaging a product of characters over the group. The simplest antisymmetric functions are alternating sums of the monomials from which all the antisymmetric functions can be constructed by taking linear combinations.

$$\zeta_{\mathbf{k}}(\phi_1,\ldots,\phi_n) = \sum_{\rho}\operatorname{sgn}(\rho)\exp(k_1\phi_1+\cdots+k_n\phi_n) \tag{2.59}$$

The sum is over all permutations ρ of the integers $\mathbf{k}$. This alternating sum can be written as a Vandemonde determinant

$$\zeta_{\mathbf{k}}(\phi_1,\ldots,\phi_n) = \begin{vmatrix} \epsilon_1^{k_1} & \epsilon_2^{k_1} & \ldots & \epsilon_n^{k_1} \\ \vdots & \vdots & & \vdots \\ \epsilon_1^{k_n} & \epsilon_2^{k_n} & \ldots & \epsilon_n^{k_n} \end{vmatrix} \tag{2.60}$$

which we will abbreviate and write as $|\epsilon_1^k,\ldots,\epsilon_n^k|$. The n rows are generated by replacing k with each of the k_i in turn. We observe that Δ itself is one of these alternating sums,

$$\Delta = \zeta_{(n-1,n-2,\ldots,1,0)} = |\epsilon_1^{n-1},\ldots,\epsilon_n^{n-1}| \tag{2.61}$$

The abbreviation $|\epsilon_1^{n-1},\ldots,\epsilon_n^{n-1}|$ will always refer to the alternating sum Δ, rather than some general alternating sum with coefficients n_1-1, n_2-1 etc. This is less elegant than Weyl's notation but will be necessary to track the different ϵ_i.

Integrating two monomials

$$\int_0^{2\pi}\cdots\int_0^{2\pi} Z_{\mathbf{k}}\,\overline{Z}_{\mathbf{k}'}\,d\phi_1\ldots d\phi_n = \delta_{k_1k_1'}\ldots\delta_{k_nk_n'} \tag{2.62}$$

From this we find the product of two of the antisymmetric functions

$$\int_0^{2\pi}\cdots\int_0^{2\pi} \zeta_{\mathbf{k}}\,\overline{\zeta}_{\mathbf{k}'}\,d\phi_1\ldots d\phi_n = n!\,\delta_{k_1k_1'}\ldots\delta_{k_nk_n'} \tag{2.63}$$

which also gives us the volume of the unitary group, $\Omega_{U(n)} = n!$.

If we take a compound character X of the group and consider the antisymmetric function $X\Delta$ it is a sum of monomials and we can order the monomials in a similar way to that used to order weights. $Z_{\mathbf{k}} > Z_{\mathbf{k}'}$ if the first non-zero difference $k_i - k_i'$ is positive. Ordering the terms of $X\Delta$ in this way we take the first term to be $c\,Z_{\mathbf{k}}$. As $X\Delta$ is antisymmetric it must also contain all the other terms in $c\,\zeta_{\mathbf{k}}$ and as any permutation of the k's is a monomial lower that $c\,Z_{\mathbf{k}}$ we know that $k_1 > k_2 > \cdots > k_n$. We can subtract $c\,\zeta_{\mathbf{k}}$ from $X\Delta$ and repeat the procedure expanding $X\Delta$ as a sum of the antisymmetric functions

$$X\Delta = c\,\zeta_{\mathbf{k}} + c'\,\zeta_{\mathbf{k}'} + \ldots \tag{2.64}$$

where $\zeta_{\mathbf{k}} > \zeta_{\mathbf{k}'} > \ldots$. Averaging $X\overline{X}$ over the group

$$\frac{1}{\Omega}\int_0^{2\pi}\cdots\int_0^{2\pi} X\overline{X}\,\Delta\overline{\Delta}\,d\phi_1\ldots d\phi_n = c^2 + c'^2 + \ldots \tag{2.65}$$

However if X is the character of an irreducible representation then the average over the group should be one, by character orthogonality. So $c = \pm 1$ and the other coefficients c' etc are zero. The irreducible characters of $U(n)$ are of the form

$$\chi(\boldsymbol{\epsilon}) = \frac{|\,\epsilon_1^k\,\ldots\,\epsilon_n^k\,|}{|\epsilon_1^{n-1}\ldots\epsilon_n^{n-1}|} \tag{2.66}$$

where $k_1 > \cdots > k_n$, this is the Weyl character formula for $U(n)$. The highest monomial in this character is

$$\epsilon_1^{f_1}\epsilon_2^{f_2}\ldots\epsilon_n^{f_n}$$

where

$$f_1 = k_1 - (n-1),\ f_2 = k_2 - (n-2),\ \dots,\ f_{n-1} = k_{n-1} - 1,\ f_n = k_n.$$

and therefore $f_1 \geq \cdots \geq f_n$.

The irreducible representations of $U(n)$ are labelled by a string of decreasing integers $(f_1, \dots, f_n)$, in the same way that we deduced from the highest weights that the representations of $SU(n)$ are labled by tableau with a string of $n-1$ decreasing integers. To use the Weyl character formula (2.66) to find the irreducible characters of $SU(n)$ we restrict the characters of $U(n)$ to elements of the $SU(n)$ subgroup. To do this we set

$$\overline{\epsilon}_n = \epsilon_1 \epsilon_2 \dots \epsilon_{n-1} \tag{2.67}$$

Irreducible representations of $U(n)$ restricted to $SU(n)$ are also irreducible representations of $SU(n)$. Let $u \in U(n)$ and $d = \operatorname{Det} u$, $(\overline{d}/d^2)^{1/n} u \in SU(n)$. If $T((\overline{d}/d^2)^{1/n} u)$ is an irreducible representation of $SU(n)$ then $d^{1/n} T((\overline{d}/d^2)^{1/n} u)$ is an irreducible representation of $U(n)$.

If we look at the character formula for $SU(n)$ with condition (2.67)

$$\chi_{\mathbf{f}} = \frac{\begin{vmatrix} \epsilon_1^{f_1+(n-1)} & \dots & \epsilon_{n-1}^{f_1+(n-1)} & (\overline{\epsilon}_1 \overline{\epsilon}_2 \dots \overline{\epsilon}_{n-1})^{f_1+(n-1)} \\ \vdots & & \vdots & \vdots \\ \epsilon_1^{f_n} & \dots & \epsilon_{n-1}^{f_n} & (\overline{\epsilon}_1 \overline{\epsilon}_2 \dots \overline{\epsilon}_{n-1})^{f_n} \end{vmatrix}}{\begin{vmatrix} \epsilon_1^{(n-1)} & \dots & \epsilon_{n-1}^{(n-1)} & (\overline{\epsilon}_1 \overline{\epsilon}_2 \dots \overline{\epsilon}_{n-1})^{(n-1)} \\ \vdots & & \vdots & \vdots \\ \epsilon_1 & \dots & \epsilon_{n-1} & (\overline{\epsilon}_1 \overline{\epsilon}_2 \dots \overline{\epsilon}_{n-1}) \\ 1 & \dots & 1 & 1 \end{vmatrix}} \tag{2.68}$$

Multiplying the first column of the determinant in the numerator by $\epsilon_1^{-f_n}$ and the last by column by $\overline{\epsilon}_1^{-f_n}$ doesn't change the determinant. Repeating this for the other columns

$$\chi_{\mathbf{f}} = \frac{\begin{vmatrix} \epsilon_1^{f_1-f_n+(n-1)} & \dots & \epsilon_{n-1}^{f_1-f_n+(n-1)} & (\overline{\epsilon}_1\overline{\epsilon}_2\dots\overline{\epsilon}_{n-1})^{f_1-f_n+(n-1)} \\ \vdots & & \vdots & \vdots \\ 1 & \dots & 1 & 1 \end{vmatrix}}{\begin{vmatrix} \epsilon_1^{(n-1)} & \dots & \epsilon_{n-1}^{(n-1)} & (\overline{\epsilon}_1\overline{\epsilon}_2\dots\overline{\epsilon}_{n-1})^{(n-1)} \\ \vdots & & \vdots & \vdots \\ \epsilon_1 & \dots & \epsilon_{n-1} & (\overline{\epsilon}_1\overline{\epsilon}_2\dots\overline{\epsilon}_{n-1}) \\ 1 & \dots & 1 & 1 \end{vmatrix}} \tag{2.69}$$

The character of $(f_1, \dots, f_n)$ is the same as $(f_1 - f_n, \dots, f_{n-1} - f_n, 0)$. This is equivalent to the statement that irreducible representations of $SU(n)$ labelled by tableau with n rows are equivalent to the representation labelled by the tableau with the columns of length n removed. We have not established that the character labelled by $(f_1, \dots, f_{n-1}, 0)$ corresponds to the same representation of $SU(n)$ that is constructed from the tableau $(f_1, \dots, f_{n-1})$ but this can be done see [53].

2.8 Characters of S_n

There is a well known formula for the irreducible characters of the symmetric group due to Frobenius, a derivation of which is found in chapter 5 of [40]. The character of an element depends on its class which we label with a partition $\boldsymbol{\omega}$ of n, where the class is of all elements with ω_1 one cycles ω_2 two cycles etc. As we saw with the characteristic units of the symmetric group the irreducible representations of S_n are also labelled by partitions of n, $\boldsymbol{\lambda} = (\lambda_1 \dots \lambda_p)$. The irreducible characters appear in the formula as coefficients in a polynomial of n variables x_r

$$(\prod_{r<s}(x_r - x_s)) \prod_{j=1\dots n} (x_1^j + \dots + x_n^j)^{\omega_j} = \sum \pm \chi^{\boldsymbol{\lambda}}_{\boldsymbol{\kappa}} x_1^{\lambda_1+p-1} x_2^{\lambda_2+p-2} \dots x_p^{\lambda_p} \tag{2.70}$$

If the left hand side is expanded once for each of the classes $\boldsymbol{\omega}$ the characters of all the irreducible representations appear as the coefficients of certain monomials in the expression. From this formula the characters of an irreducible representation of S_n can be computed.

2.9 The Littlewood-Richardson theorem

The Littlewood-Richardson theorem is a combinatorial rule for computing the coefficients in the decomposition of a product of primitive characteristic units of the symmetric group. Let $A^{\boldsymbol{\lambda}}_{S_m}$ be a primitive characteristic unit of S_m acting on m symbols constructed from a tableau $[\boldsymbol{\lambda}]$ and $A^{\boldsymbol{\eta}}_{S_n}$ be a primitive characteristic unit acting on n different symbols. We know that $S_m \times S_n$ is a subgroup of S_{m+n}. The product of the two primitive characteristic units $A^{\boldsymbol{\lambda}}_{S_m} A^{\boldsymbol{\eta}}_{S_n}$ will also be idempotent and so a characteristic unit of S_{m+n}. In general it will not correspond to an irreducible representation of S_{m+n} but will be a sum of primitive characteristic units of S_{m+n}

$$A^{\boldsymbol{\lambda}}_{S_m} A^{\boldsymbol{\eta}}_{S_n} = \sum_{\boldsymbol{\kappa}} Y^{\boldsymbol{\kappa}}_{\boldsymbol{\lambda\eta}} A^{\boldsymbol{\kappa}}_{S_{m+n}} \tag{2.71}$$

It is these coefficients $Y^{\boldsymbol{\kappa}}_{\boldsymbol{\lambda\eta}}$ that we want to compute.

The Littlewood Richardson Theorem 2.9.1.

To every tableau which can be constructed according to the following rule there corresponds a primitive characteristic unit $A^{\boldsymbol{\kappa}}_{S_{m+n}}$ in the decomposition of the product $A^{\boldsymbol{\lambda}}_{S_m} A^{\boldsymbol{\eta}}_{S_n}$, and this decomposition is complete.

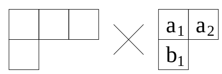
LR_1: Take the tableau $[\boldsymbol{\lambda}]$ intact and add to it the symbols in the first row of $[\boldsymbol{\eta}]$ to make a new tableau without changing the order of the symbols. After the addition no two added symbols may be in the same column. Next add the remaining rows of $[\boldsymbol{\eta}]$ in succession according to the same rule.

LR_2: The only allowed additions are those where each symbol in $[\boldsymbol{\eta}]$ is placed in a later row than the symbol in the same column from the preceding row of $[\boldsymbol{\eta}]$.

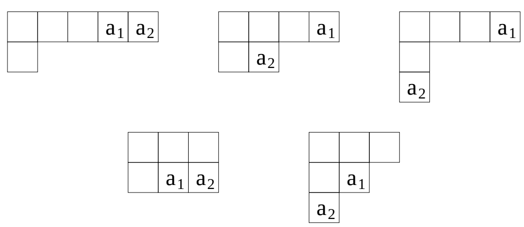
These rules were originally proposed by Littlewood and Richardson in 1934 [41], however the subsequent proofs in [47] and [40] are not complete, the first correct proofs appeared in the 70's some forty years later. One version of the complete proof is given by Macdonald in [42].

2.9.1 Multiplying Young tableau

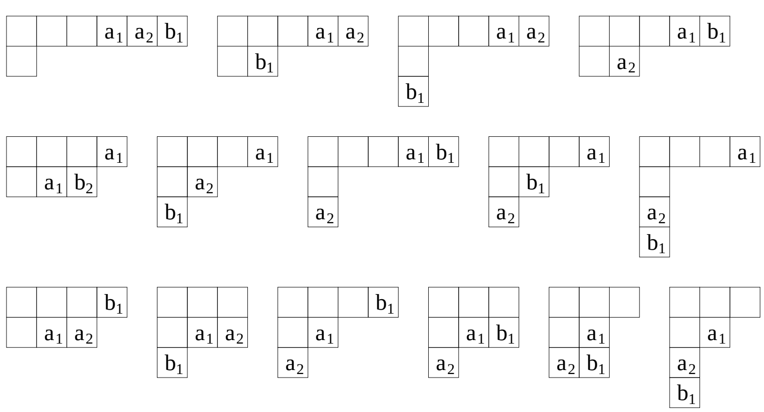
The Littlewood-Richardson rules (LR) can be more simply stated as a procedure for multiplying two tableau one of shape $\boldsymbol{\lambda}$ and the other $\boldsymbol{\eta}$. Take the graph $\boldsymbol{\eta}$, without loss of generality it can be assumed $|\boldsymbol{\eta}| \leq |\boldsymbol{\lambda}|$, and fill the boxes on the first row with a's, the second row with b's etc. So if $\boldsymbol{\lambda} = (3, 1)$, $\boldsymbol{\eta} = (2, 1)$ we have



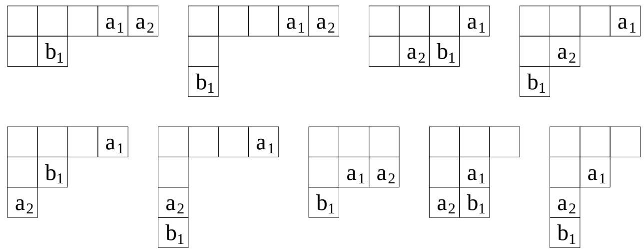
Add the a's to $\boldsymbol{\lambda}$ in any way which makes a new graph with a maximum of one a in each column. In our example there are five possibilities



Then repeat the procedure for each of the other rows in turn with the added condition that counting right to left and top to bottom the number of a's $\geq$ the number of b's $\geq$ the number of c's $\geq \ldots$. So in our example adding the single b in all possible ways to get a graph we find



If we take the first of these tableau and count the number of a's and b's from the right we start with one b and no a's in the right-hand column. This doesn't agree with the condition that the number of a's always be greater than or equal to the number of b's so we discard this tableau. Checking the other tableau similarly and counting top to bottom as well we finish with a set of nine tableau.



Within this set of tableau the graph $(4,2,1)$ is repeated twice with different arrangements of the letters. In terms of characteristic units we have two primitive characteristic units of S_{m+n} which both correspond to the same irreducible representation, two equivalent primitive characteristic units. The number of graphs $\boldsymbol{\kappa}$ in the product of the two tableau is the coefficient $Y^{\boldsymbol{\kappa}}_{\boldsymbol{\lambda\eta}}$, so in our example

$$Y^{(4,2,1)}_{(3,1)(2,1)} = 2$$

2.9.2 Applying the Littlewood-Richardson theorem to representations

As the primitive characteristic units project onto irreducible representations of the symmetric groups the Littlewood-Richardson theorem is also a theorem about the decomposition of a representation of S_{m+n} into irreducible representations of the $S_m \times S_n$ subgroup.

Consider any group G of order Ω_G with a subgroup H of order Ω_H. An irreducible representation of G restricts to a representation of H. If the irreducible characters of G are X_i and the irreducible characters of H are χ_j then for $h \in H$

$$X_i(h) = \sum_j c_{ij}\chi_j(h) \tag{2.72}$$

c_{ij} is the multiplicity of the representation j of H when the representation i of G is restricted to H.

The character of a group is a function of the classes of the group. Let α be a class of G of order Ω_G^α, Ω_H^α of these elements are in H. Elements in different classes of G must be in different classes of H but a class of G can break up into several classes of H. We label the classes of H which are in the class α of G with $\alpha_1, \ldots, \alpha_r$. Using the character orthogonality relation (2.18) we can find the coefficients c_{ik}

$$c_{ik} = \frac{1}{\Omega_H} \sum_{\alpha_j} \Omega_H^{\alpha_j} X_i(\alpha) \overline{\chi}_k(\alpha_j) \tag{2.73}$$

The sum is over all classes of H. With a formula for the coefficients c_{ik} we can investigate the composite character of G that they define.

$$\begin{aligned} \sum_i c_{ik} \overline{X}_i(\beta) &= \frac{1}{\Omega_H} \sum_{\alpha_j} \Omega_H^{\alpha_j} \overline{\chi}_k(\alpha_j) \sum_i X_i(\alpha) \overline{X}_i(\beta) \\ &= \frac{1}{\Omega_H} \sum_{\beta_j} \frac{\Omega_H^{\beta_j}}{\Omega_G^{\beta}} \overline{\chi}_k(\beta_j) \Omega_G \end{aligned} \tag{2.74}$$

where we applied the orthogonality relation (2.19). This can be rearranged into a second formula relating the characters of a group and its subgroup, so we have

$$\begin{aligned} X_i(h) &= \sum_j c_{ij} \chi_j(h) \\ \sum_i c_{ij} X_i(\beta) &= \sum_{\beta_k} \frac{\Omega_G \Omega_H^{\beta_k}}{\Omega_H \Omega_G^{\beta}} \chi_j(\beta_k) \end{aligned} \tag{2.75}$$

The multiplicity of the representation j of H when the representation i of G is restricted to H is the same as the multiplicity of the representation i of G when the representation j of H is used to induce a representation of G.

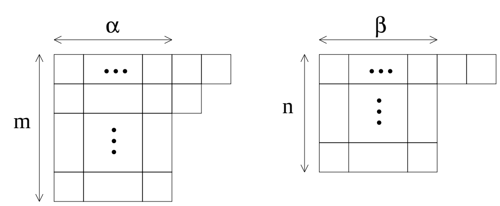
Returning to characteristic units of the symmetric group, the Littlewood-Richardson theorem gives a procedure for calculating the coefficients of the primitive characters of S_{m+n} from a primitive character of the subgroup $S_m \times S_n$, (2.71). These are the coefficients of the induced representation in (2.75) so the same coefficients from the Littlewood-Richardson theorem also give the decomposition of a representation of S_{m+n} into irreducible representations of $S_m \times S_n$.

2.9.3 Decomposing representations of $SU(m+n)$ into representations of $SU(m) \times SU(n)$

In section 2.6.4 we discussed how characteristic units of S_n are used to construct irreducible representations of $SU(n)$. In [33] and [37] the Littlewood-Richardson theorem for multiplying tableau is used to decompose a representation of $SU(m+n)$ into irreducible representations of its $SU(m) \times SU(n)$ subgroup. We know that the multiplicity of an irreducible representation $T_j(SU(m) \times SU(n))$ when an irreducible representation $R_i(SU(m+n))$ is restricted to $SU(m) \times SU(n)$ is equal to the number of representations T_j induced by a representation R_i of the subgroup.

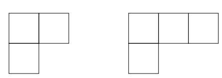
A representation of $SU(m) \times SU(n)$ is constructed from two characteristic units, one of S_q and another of S_p. The characteristic unit of S_q (respectively S_p) act on the tensor product of q (p) basis vectors of the defining representations of $SU(m)$ (respectively $SU(n)$). The product of the two primitive characteristic units is a characteristic unit of S_{q+p} and induces a representation of $SU(m+n)$. The Littlewood-Richardson theorem gives the number of primitive characteristic units of S_{q+p} in the decomposition of the product of the primitive characteristic units of S_q and S_p. Each primitive characteristic unit generates an irreducible representations of $SU(m+n)$. The number of irreducible representations of $SU(m+n)$ induced by the representation of $SU(m) \times SU(n)$ is the same as the multiplicity of the representation of $SU(m) \times SU(n)$ in the decomposition of the chosen representation of $SU(m+n)$.

Itzykson and Nauenberg [37] point out that this simple situation is complicated slightly by the use of multiple graphs to label a single representation of $SU(n)$. All the irreducible representations of $SU(m) \times SU(n)$ are labelled by two graphs of $m-1$ and $n-1$ rows respectively but without changing the representation we can add α columns of m boxes to the first graph and β columns of n boxes to the second. For example the representation $(2,1)(2)$ corresponds to the graphs

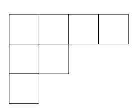


Applying the Littlewood-Richardson theorem we multiply the two tableau producing tableau which label irreducible representations of $SU(m+n)$. Counting the number of tableau of a particular shape gives the multiplicity of the irreducible representations of $SU(m) \times SU(n)$ in the decomposition of the representation of $SU(n+m)$ labelled by a tableau of that shape. Tableau with columns of $m+n$ boxes can be discarded as the results for these representations of $SU(m+n)$ can be obtained by considering the tableau with the columns of length $m+n$ removed.

Let us look at a simple example. We will find the multiplicity of the representation $(2,1)(2)$ of $SU(3) \times SU(2)$ when the representation $(4,2,1)$ of $SU(5)$ is restricted to the $SU(3) \times SU(2)$ subgroup. The representation $(4,2,1)$ is labelled by a tableau of seven boxes. For this to be a result of the multiplication of two tableau the two tableau must have a total of seven boxes between them. To the tableau $(2,1)$ we can add α columns of three boxes without changing the representation and we can add β columns of two boxes to (2). For the two tableau to consist of seven boxes in total we must take $\alpha = 0$ and $\beta = 1$. We want to find the product of the tableau



This is the tableau multiplication done in section 2.9.1. The graph



appears twice in the result so the representation $(4,2,1)$ of $SU(5)$ when restricted to the $SU(3) \times SU(2)$ subgroup contains two copies of the $(2,1)(2)$ irreducible representation of the subgroup. It is this procedure that will be extended in chapter 4 to provide a tableau method achieving a similar decomposition for a subgroup of $SU(4)$.

We have now reviewed the representations and characters of the symmetric and unitary groups and discussed the relationship between them. We have also discussed results relating to the decomposition of a representation of a group restricted to a subgroup culminating in the Littlewood-Richardson theorem which provides a method of evaluating the multiplicity of representations both of subgroups of the symmetric and the unitary groups. This should provide the necessary tools for the subsequent investigation.

Chapter 3

Quantum indistinguishability

Berry and Robbins (BR) in a series of papers [8], [9] and [10] have proposed an alternative formulation of non-relativistic quantum mechanics in which the spin-statistics theorem is derived from the properties of a position-dependent spin basis of the wavefunctions. In this chapter we will review this construction in order to demonstrate the underlying group-theoretical properties. The particular use of the Schwinger operators in [8] can then be seen as a choice of a certain set of representations of the groups. The subsequent chapters will involve deriving the properties of this construction for general representations.

I will not present the entire scope of their work here. In particular the discussions of the relationship between the construction, relativity and the invariance of the system under Lorentz or Galilean transformations has been omitted along with the extension to particles with additional quantum properties such as colour, strangeness or isospin (These are discussed in [9]).

3.1 Introduction

The construction of Berry and Robbins suggests an elementary non-relativistic basis for the spin-statistics theorem. As quantum mechanics alone is insufficient to derive a spin-statistics connection it is necessary to include additional physical postulates. To be accepted as an explanation of spin-statistics these additional requirements should be more transparent than the symmetrisation postulate itself. Berry and Robbins suggest that the spin-statistics connection could be a consequence of two additions to normal quantum mechanics, the correct incorporation of the indistinguishability of identical particles (so that the space of wavefunctions has built into it the indistinguishability of states related by the exchange of particles positions and spins) and the singlevaluedness of wavefunctions on this space.

The construction is reminiscent one of the classical belt tricks that Feynman found indicative of the spin-statistics connection, see figure 1.2. In this model fermions are seen as tethered objects where exchange introduces a twist in the ribbon connecting them whilst bosons correspond to the ordinary untethered objects. We will see that the construction avoids the objection that elementary particles have no ribbon-like topological marker as the spin basis itself records the exchange of fermions without a classical ribbon.

3.2 The position-dependent spin basis

The wavefunction of a system of n particles with spin s would normally be expanded on a spin basis $|\mathbf{M}\rangle$ where the basis vectors are labelled by the z-components of the n spins, $\mathbf{M} \equiv \{m_1, m_2, \ldots, m_n\}$.

$$|\Psi(\mathbf{R})\rangle = \sum \psi_M(\mathbf{R})|\mathbf{M}\rangle \tag{3.1}$$

where $\mathbf{R} \equiv \{\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n\}$ is a point in the configuration space of the n particles, the space $\mathbb{R}^{3n}$ with coincident points removed so no two particles can occupy the same position. In this description the identical particles are still identified by their labels so, for example, if we exchange particles 1 and 2 this is a different point in the configuration space. In order to make the particles truly indistinguishable permuted configurations of particles will be identified and we will insist that the wavefunction be singlevalued on this new configuration space. So if ρ is an element of S_n

$$|\Psi(\mathbf{R})\rangle = |\Psi(\rho\mathbf{R})\rangle \tag{3.2}$$

The permuted configuration is $\rho\mathbf{R} = \{\mathbf{r}_{\rho(1)}, \ldots \mathbf{r}_{\rho(n)}\}$. To exchange the particles, rather than just the position labels, we must exchange the spins of the particles when we exchange the positions. In BR this is done using a position-dependent unitary transformation $U(\mathbf{R})$ to generate a position-dependent spin basis,

$$|\mathbf{M}(\mathbf{R})\rangle = U(\mathbf{R})|\mathbf{M}\rangle \tag{3.3}$$

This position-dependent spin basis $|\mathbf{M}(\mathbf{R})\rangle$ is required to have certain properties.

3.2.1 The basis depends smoothly on $\mathbf{R}$.

3.2.2 There is a single state for all permutations of particles (this excludes parastatistics),

$$|\rho\,\mathbf{M}(\rho\,\mathbf{R})\rangle = e^{i\alpha_\rho(\mathbf{R})}|\mathbf{M}(\mathbf{R})\rangle \tag{3.4}$$

ρ is a permutation of the n particles with their spins,

$$\rho(\mathbf{M}) = \{m_{\rho(1)}, \ldots, m_{\rho(n)}\}$$

3.2.3 Spins are parallel-transported.

$$\langle\mathbf{M}'(\mathbf{R})|\nabla\mathbf{M}(\mathbf{R})\rangle = 0 \tag{3.5}$$

$\mathbf{M}'$ and $\mathbf{M}$ are arbitrary sets of spin quantum numbers. This requirement ensures that their are no local changes in phase associated with travelling around a contractible closed loop.

By combining the second and third conditions we can show that the position-dependent spin basis must transform according to the trivial or alternating representation of the permutation group. First we will take σ to be an exchange of two of the particles keeping the rest fixed. Applying the condition (3.4) twice we find that

$$\begin{aligned} |\mathbf{M}(\mathbf{R})\rangle &= |\sigma^2\,\mathbf{M}(\sigma^2\,\mathbf{R})\rangle = e^{i\alpha_\sigma(\sigma\mathbf{R})}|\sigma\,\mathbf{M}(\sigma\,\mathbf{R})\rangle \\ &= e^{i(\alpha_\sigma(\sigma\mathbf{R})+\alpha_\sigma(\mathbf{R}))}|\mathbf{M}(\mathbf{R})\rangle \end{aligned} \tag{3.6}$$

The general solution of this is

$$\alpha_\sigma(\mathbf{R}) = k\pi + \beta_\sigma(\mathbf{R}) \tag{3.7}$$

where k is an integer and

$$\beta_\sigma(\sigma\mathbf{R}) = -\beta_\sigma(\mathbf{R})$$

Using (3.4) and the result (3.7) we can write

$$\begin{aligned}\langle\sigma\mathbf{M}'(\sigma\mathbf{R})|\nabla\sigma\mathbf{M}(\sigma\mathbf{R})\rangle &= \langle\mathbf{M}'(\mathbf{R})|\nabla\mathbf{M}(\mathbf{R})\rangle + i(\nabla\beta_\sigma(\mathbf{R}))\,\langle\mathbf{M}'(\mathbf{R})|\mathbf{M}(\mathbf{R})\rangle \\ &= \langle\mathbf{M}'(\mathbf{R})|\nabla\mathbf{M}(\mathbf{R})\rangle + i(\nabla\beta_\sigma(\mathbf{R}))\,\delta_{\mathbf{M}\mathbf{M}'} \end{aligned} \tag{3.8}$$

Applying the parallel-transport condition we have

$$(\nabla\beta_\sigma(\mathbf{R}))\,\delta_{\mathbf{M}\mathbf{M}'} = 0 \tag{3.9}$$

To satisfy (3.9) for all $\mathbf{R}$, $\mathbf{M}$ and $\mathbf{M}'$ requires $\beta_\sigma(\mathbf{R})$ to be a constant. As $\beta_\sigma(\mathbf{R})$ changes sign under odd permutations it must be identically zero. We have the result

$$|\sigma\mathbf{M}(\sigma\mathbf{R})\rangle = (-1)^k|\mathbf{M}(\mathbf{R})\rangle \tag{3.10}$$

In deriving (3.10) we assumed that the permutation σ was an exchange of two particles. However as all permutations can be constructed from a product of two-cycles this can be rewritten for a general permutation ρ

$$|\rho\mathbf{M}(\rho\mathbf{R})\rangle = (-1)^{k\,\mathrm{sgn}(\rho)}|\mathbf{M}(\mathbf{R})\rangle \tag{3.11}$$

If k is even the states of the position-dependent spin basis transform according to the trivial representation of the permutation group. If k is odd the states transform according to the alternating representation.

In [8] BR provide a construction of $U(\mathbf{R})$ for which they show that

$$k = 2s \tag{3.12}$$

The construction for n particles depends on the solution of a topological problem I will describe later. In the two particle case the solution of this problem is simple and the construction can be easily explained. Before looking at the explicit construction of $U(\mathbf{R})$ we will see how taking $k = 2s$ leads to the spin-statistics connection.

3.3 Quantum mechanics in the position-dependent spin basis

First we assert the condition that wavefunctions on the position-dependent basis are singlevalued.

$$|\Psi(\rho\mathbf{R})\rangle = |\Psi(\mathbf{R})\rangle \tag{3.13}$$

From (3.11) and (3.12) we know that

$$\begin{aligned} |\Psi(\rho\mathbf{R})\rangle &= \sum_{\mathbf{M}} \psi_{\mathbf{M}}(\rho\mathbf{R})(-1)^{2s\,\mathrm{sgn}(\rho)}|\rho^{-1}\mathbf{M}(\mathbf{R})\rangle \\ &= \sum_{\mathbf{M}} \psi_{\rho\mathbf{M}}(\rho\mathbf{R})(-1)^{2s\,\mathrm{sgn}(\rho)}|\mathbf{M}(\mathbf{R})\rangle \end{aligned} \tag{3.14}$$

So for the components of the wavefunction we have

$$\psi_{\rho\mathbf{M}}(\rho\mathbf{R}) = (-1)^{2s\,\mathrm{sgn}(\rho)}\psi_{\mathbf{M}}(\mathbf{R}) \tag{3.15}$$

This has the form of the spin-statistics relation and to demonstrate that it is equivalent we show that these coefficients obeys the same Schrödinger equation as the coefficients in the usual spin basis.

An operator A in the position dependent spin basis is

$$A(\mathbf{R}) = U(\mathbf{R})AU^{\dagger}(\mathbf{R}) \tag{3.16}$$

In this way the operators in the fixed- and position-dependent spin bases obey the same commutation relations. If we compare the action of A in the two bases

$$\begin{aligned} \langle\mathbf{M}(\mathbf{R})|A(\mathbf{R})|\Psi(\mathbf{R})\rangle &= \sum_{\mathbf{M}'}\langle\mathbf{M}|U^{\dagger}(\mathbf{R})U(\mathbf{R})AU^{\dagger}(\mathbf{R})\psi_{\mathbf{M}'}(\mathbf{R})U(\mathbf{R})|\mathbf{M}'\rangle \\ &= \sum_{\mathbf{M}'}\langle\mathbf{M}|A\psi_{\mathbf{M}'}(\mathbf{R})|\mathbf{M}'\rangle \\ &= \langle\mathbf{M}|A|\Psi\rangle \end{aligned} \tag{3.17}$$

which is the same as the equation for the action of the operator A in the fixed-spin basis. By defining the operators on the position-dependent spin basis in this way all the usual properties of non-relativistic quantum mechanics remain unchanged. So for a choice of $U(\mathbf{R})$ with the required properties and where $k = 2s$ the wavefunctions are required to obey the spin-statistics theorem while all the usual properties of quantum mechanics are maintained.

3.4 The Schwinger representation of spin

In the Schwinger representation of spin a pair of harmonic oscillators with creation and annihilation operators $a^\dagger$, a and $b^\dagger$, b are assigned to each spinning particle. The spin operators are defined using the Pauli matrices

$$\mathbf{S} = \frac{1}{2}\begin{pmatrix} a^\dagger & b^\dagger \end{pmatrix} \boldsymbol{\sigma} \begin{pmatrix} a \\ b \end{pmatrix} \tag{3.18}$$

$\mathbf{S}$ is a vector of spin operators defined by the vector of Pauli matrices $\boldsymbol{\sigma}$. In the Schwinger representation states are labelled by the number of quanta in each harmonic oscillator. From (3.18)

$$S_z = \frac{1}{2}(a^\dagger a - b^\dagger b)$$

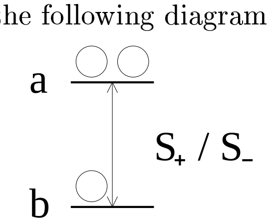
As $a^\dagger a$ is the number operator for the "a" oscillator the z-component of spin for a state is

$$m = \frac{1}{2}(n_a - n_b)$$

where n_a is the number of quanta in oscillator a. Similarly the total spin of the particle is

$$s = \frac{1}{2}(n_a + n_b)$$

We can think of the state as being represented by $2s$ quanta split between the two oscillators. The spin raising and lowering operators move a quantum from one oscillator to the other, changing the z-component of spin by one whilst keeping the total spin the same. For example with three quanta, spin $3/2$, a state with $m_z = 1/2$ is described schematically by the following diagram.



To construct spin states for the n spins we use n pairs of oscillators,

$$a_1, b_1, \ldots, a_n, b_n$$

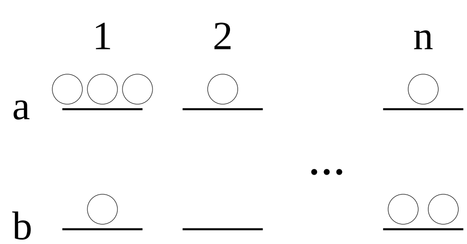
In order for the operator $U(\mathbf{R})$ to exchange spins, BR define an exchange algebra of operators $\mathbf{E}^{ij}$ which move quanta between the pairs of spin oscillators.

$$\mathbf{E}_a^{ij} = \frac{1}{2} \begin{pmatrix} a_i^\dagger & a_j^\dagger \end{pmatrix} \boldsymbol{\sigma} \begin{pmatrix} a_i \\ a_j \end{pmatrix} \tag{3.19}$$

Given a pair of oscillators a_i, a_j this defines a vector of three exchange operators $\mathbf{E}_a^{ij}$ as in the case of the spin operators. A second set of operators $\mathbf{E}_b^{ij}$ is defined similarly then

$$\mathbf{E}^{ij} = \mathbf{E}_a^{ij} + \mathbf{E}_b^{ij} \tag{3.20}$$

Defining the exchange algebra in this way an operator E_+^{ij} moves a quanta from oscillator a_j to a_i and one from b_j to b_i. We can see that including these operations the total spin of the individual particles is no longer fixed but the total number of quanta in the whole system of oscillators is still constant. Schematically a general spin state can be represented as a distribution of $2sn$ quanta between the oscillators.



The eigenvalue of such a state with respect to the operator E_z^{ij} is e_{ij}.

$$\begin{aligned} e_{ij} &= \frac{1}{2}(n_{a_i} + n_{b_i} - n_{a_j} - n_{b_j}) \\ &= s_i - s_j \end{aligned} \tag{3.21}$$

States can be labelled by the eigenvalues of the z components of the exchange algebra e_{ij} and the z components of the spins m_j instead of the occupation numbers n_{ai}, n_{bi} of the oscillators.

$$|\boldsymbol{\mu}\rangle = |e_{12}, e_{13}, \ldots, e_{n-1\,n}, m_1, \ldots, m_n\rangle \tag{3.22}$$

For states $|\boldsymbol{\mu}\rangle$ where the spins of all the particles are equal, the eigenvalues e_{ij} of the exchange algebra are all zero, see equation (3.21), and we can return to our original label of the state $|\mathbf{M}\rangle$.

3.5 $U(\mathbf{R})$ for two particles

In [8] BR construct a unitary operator $U(\mathbf{R})$ for two particles using the Schwinger representation. For two particles the position-dependent basis is a function of the relative position of the particles, $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$. Then exchanging the particles changes $\mathbf{r}$ to $-\mathbf{r}$. The spin basis is $|M\rangle = |m_1, m_2\rangle$ where the state with the spins exchanged is denoted $|\overline{M}\rangle = |m_2, m_1\rangle$. The position-dependent spin basis is generated by $U(\mathbf{r})$ so

$$|M(\mathbf{r})\rangle = U(\mathbf{r})|M\rangle \tag{3.23}$$

From equation (3.11) the exchange requirement for two particles is

$$|\overline{M}(-\mathbf{r})\rangle = (-1)^k |M(\mathbf{r})\rangle \tag{3.24}$$

where $(-1)^k$ is the exchange sign for the position-dependent basis generated by $U(\mathbf{r})$.

Using the Schwinger representation of the spin basis we can write the exchanged state $|\overline{M}\rangle$ in terms of spin state $|M\rangle$. To achieve this we define an operator generated by the element E_y^{12} of the exchange algebra,

$$Y = \exp(-i\pi E_y^{12}) \tag{3.25}$$

A state $|\mu\rangle$ of the Schwinger representation is formed by applying creation operators to the ground state. A state with n_{a_1} quanta in the oscillator a_1 etc is written

$$|\mu\rangle = (a_1^\dagger)^{n_{a_1}} (a_2^\dagger)^{n_{a_2}} (b_1^\dagger)^{n_{b_1}} (b_2^\dagger)^{n_{b_2}} |0\rangle \tag{3.26}$$

We can split the operator Y into operators Y_a and Y_b operating on the a and b oscillators respectively. Y_a and Y_b commute and the operator Y is the product $Y_a Y_b$.

$$Y_a = \exp(-i\pi E_{ay}^{12}) \tag{3.27}$$

The operator Y_a induces a transformation on the vector of creation operators

$$\begin{aligned}
\begin{pmatrix} a_1'^\dagger & a_2'^\dagger \end{pmatrix} &= Y_a \begin{pmatrix} a_1^\dagger & a_2^\dagger \end{pmatrix} Y_a^\dagger \\
&= \begin{pmatrix} a_1^\dagger & a_2^\dagger \end{pmatrix} \exp(-i\frac{\pi}{2}\sigma_y) \\
&= \begin{pmatrix} a_1^\dagger & a_2^\dagger \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} a_2^\dagger & -a_1^\dagger \end{pmatrix}
\end{aligned} \tag{3.28}$$

Y_b induces a similar transformation in the operators $b_1^\dagger$ and $b_2^\dagger$. Using these results we can write the action of Y on the state $|\mu\rangle$

$$Y|\mu\rangle = (-1)^{n_{a2}+n_{b2}}(a_1^\dagger)^{n_{a2}}(a_2^\dagger)^{n_{a1}}(b_1^\dagger)^{n_{b2}}(b_2^\dagger)^{n_{b1}}|0\rangle \tag{3.29}$$

For a state $|M\rangle$ where $s_1 = s_2 = s$ we find that

$$Y|M\rangle = (-1)^{2s}|\overline{M}\rangle \tag{3.30}$$

We can use this exchanged state to investigate properties of the position-dependent basis.

We will assume that $U(\mathbf{r})$ is generated by the algebra of exchange operators so that

$$U(\mathbf{r}) = \exp(-i\mathbf{c}(\mathbf{r}).\mathbf{E}) \tag{3.31}$$

Starting from the exchange requirement (3.24) we know that

$$U(-\mathbf{r})|\overline{M}\rangle = (-1)^k U(\mathbf{r})|M\rangle \tag{3.32}$$

Using the operator Y to rewrite $|\overline{M}\rangle$ from equation (3.30) we have

$$\begin{aligned}
U(-\mathbf{r})Y|M\rangle &= (-1)^{k-2s}U(\mathbf{r})|M\rangle \\
U^\dagger(\mathbf{r})U(-\mathbf{r})Y|M\rangle &= (-1)^{k-2s}|M\rangle
\end{aligned} \tag{3.33}$$

We can now define a new operator $V(\mathbf{r})$ where

$$V(\mathbf{r}) \equiv U^\dagger(\mathbf{r})U(-\mathbf{r})Y \tag{3.34}$$

$V(\mathbf{r})$ is also generated by the exchange algebra and

$$V(\mathbf{r})|M\rangle = (-1)^{k-2s}|M\rangle \quad \text{for all } |M\rangle \tag{3.35}$$

This implies that $V(\mathbf{r})$ is diagonal in the $|M\rangle$ basis. An operator

$$\exp(-id(\mathbf{r})E_z^{12})$$

generated by E_z^{12} is diagonal in any basis and next we will show that $V(\mathbf{r})$ is such an operator.

In the $|M\rangle$ basis $V(\mathbf{r})$ must commute with any operator generated by E_z^{12}.

$$V(\mathbf{r})\exp(-id(\mathbf{r})E_z^{12})V^\dagger(\mathbf{r}) = \exp(-id(\mathbf{r})E_z^{12}) \tag{3.36}$$

This is equivalent to

$$\exp(-id(\mathbf{r})\{V(\mathbf{r})E_z^{12}V^\dagger(\mathbf{r})\}) = \exp(-id(\mathbf{r})E_z^{12}) \tag{3.37}$$

which implies that

$$V(\mathbf{r})E_z^{12}V^\dagger(\mathbf{r}) = E_z^{12} \tag{3.38}$$

in the $|M\rangle$ basis. To any element generated by the exchange algebra

$$\exp(-i\mathbf{c}(\mathbf{r}).\mathbf{E}^{12})$$

we can associate a rotation matrix $\mathcal{R}_{\hat{\mathbf{c}}}(|\mathbf{c}|)$, a rotation about the axis $\hat{\mathbf{c}}$ by an angle $|\mathbf{c}|$. Then using the rotation matrix $\mathcal{R}^V$ associated to $V(\mathbf{r})$ from (3.38) we have

$$\mathcal{R}^V\hat{z} = \hat{z} \tag{3.39}$$

The $\hat{z}$ direction is invariant under the spatial rotation associated to $V(\mathbf{r})$ and consequently it must correspond to a rotation about the $\hat{z}$ axis.

$$V(\mathbf{r}) = \exp(-id(\mathbf{r})E_z) \tag{3.40}$$

3.5.1 The exchange sign is independent of $U(\mathbf{r})$

Using these results for $V(\mathbf{r})$ we can see that the exchange sign $(-1)^k$ is independent of the particular form of $U(\mathbf{r})$. From equation (3.21) we know that the spin vectors are null states of E_z^{12}.

$$E_z^{12}|M\rangle = 0 \tag{3.41}$$

As $V(\mathbf{r})$ is generated by E_z^{12} this implies

$$V(\mathbf{r})|M\rangle = |M\rangle \tag{3.42}$$

Comparing this to equation (3.35) we see that

$$k = 2s \tag{3.43}$$

The exchange sign $(-1)^k$ is that required to produce the correct spin-statistics connection for any operator $U(\mathbf{r})$ with the required properties. The exchange sign is topological.

3.5.2 Constructing $U(\mathbf{r})$

These results also suggest how to construct the operator $U(\mathbf{r})$. We saw previously that any operator generated by E_z^{12} does not effect the spin states $|M\rangle$.

$$\exp(i\alpha(\mathbf{r})E_z^{12})|M\rangle = |M\rangle \tag{3.44}$$

If we define a new unitary operator $U'(\mathbf{r})$ where

$$U'(\mathbf{r}) = U(\mathbf{r})\exp(i\alpha(\mathbf{r})E_z^{12}) \tag{3.45}$$

then

$$|M(\mathbf{r})\rangle = U(\mathbf{r})|M\rangle = U'(\mathbf{r})|M\rangle \tag{3.46}$$

The same position-dependent basis is generated by both U and U'.

To determine $U(\mathbf{r})$ it is sufficient to determine the action of the associated spatial rotation $\mathcal{R}^U(\mathbf{r})$ on the $\hat{z}$ axis. From the definition (3.34) of $V(\mathbf{r})$ we have

$$(\mathcal{R}^U(\mathbf{r}))^{-1}\mathcal{R}^U(-\mathbf{r})\mathcal{R}_{\hat{y}}(\pi)\hat{z} = \hat{z} \tag{3.47}$$

where $\mathcal{R}_{\hat{y}}(\pi)$ is a rotation by π about the $\hat{y}$ axis, the rotation associated to the operator Y. This condition on the rotations reduces too

$$\mathcal{R}^U(\mathbf{r})\hat{z} = -\mathcal{R}^U(-\mathbf{r})\hat{z} \tag{3.48}$$

To simplify this further we can define $\hat{e}(\mathbf{r}) = \mathcal{R}^U(\mathbf{r})\hat{z}$ then

$$\hat{e}(-\mathbf{r}) = -\hat{e}(\mathbf{r}) \tag{3.49}$$

or $\hat{e}(\mathbf{r})$ is odd. A simple solution with this property is to take $\hat{e} = \hat{\mathbf{r}}$ so

$$\mathcal{R}^U(\mathbf{r})\hat{z} = \hat{\mathbf{r}} \tag{3.50}$$

So one example of the operator $U(\mathbf{r})$ which generates the position-dependent spin basis is

$$U(\mathbf{r}) = \exp(-i\theta\hat{\mathbf{n}}.\mathbf{E}^{12}) \tag{3.51}$$

where $\mathcal{R}_{\hat{\mathbf{n}}}(\theta)\hat{z} = \hat{\mathbf{r}}$.

3.5.3 Smoothness of $|M(\mathbf{r})\rangle$

The choice of $U(\mathbf{r})$ in (3.51) is smooth everywhere except for the south pole. Therefore the position dependent basis $|M(\mathbf{r})\rangle$ must also be smooth except possibly at the south pole. We can choose a second operator $U'(\mathbf{r})$ related to $U(\mathbf{r})$ by a rotation about the $\hat{z}$ axis as in equation (3.45). $U'(\mathbf{r})$ will be an alternative exchange rotation first from $\hat{\mathbf{z}}$ to $-\hat{\mathbf{z}}$ then from $-\hat{\mathbf{z}}$ to $\mathbf{r}$

$$U'(\mathbf{r}) = \exp(-i(\pi-\theta)\hat{\mathbf{n}}.\mathbf{E}^{12})\exp(-i\pi E_y^{12}) \tag{3.52}$$

$U'(\mathbf{r})$ is clearly smooth near the south pole so the position-dependent basis $|M(\mathbf{r})\rangle$ is also smooth there. As both $U(\mathbf{r})$ and $U'(\mathbf{r})$ generate the same position-dependent spin basis, the basis is smooth everywhere.

3.5.4 Parallel-transport of $|M(\mathbf{r})\rangle$

The parallel-transport condition (3.5) is

$$\langle M'(\mathbf{r})|\nabla M(\mathbf{r})\rangle = \langle M'|U^\dagger(\mathbf{r})\nabla U(\mathbf{r})|M\rangle = 0 \tag{3.53}$$

$U(\mathbf{r})$ and $U(\mathbf{r}+d\mathbf{r})$ are infinitesimally different exchange rotations and so differ only by a linear combination of the elements of the exchange algebra.

$$U^\dagger(\mathbf{r})\nabla U(\mathbf{r}) = \alpha(\mathbf{r})E_z^{12} + \beta(\mathbf{r})E_+^{12} + \gamma(\mathbf{r})E_-^{12} \tag{3.54}$$

Therefore

$$\begin{gathered}\langle M'(\mathbf{r})|\nabla M(\mathbf{r})\rangle = \\ \alpha(\mathbf{r})\langle M'|E_z^{12}|M\rangle + \beta(\mathbf{r})\langle M'|E_+^{12}|M\rangle + \gamma(\mathbf{r})\langle M'|E_-^{12}|M\rangle\end{gathered} \tag{3.55}$$

The physical spin states $|M\rangle$ of the Schwinger representation are null states of E_z^{12},

$$E_z^{12}|M\rangle = 0 \tag{3.56}$$

So the "α" term in (3.55) vanishes. As E_+^{12} and E_-^{12} operators raise or lower the total spin of one of the particles the "β" and "γ" terms are inner products of a spin state (s, s) with a state $(s \pm 1/2, s \mp 1/2)$. These also vanish which demonstrates that the position-dependent basis parallel-transports spins.

3.6 The Schwinger representation of $SU(2n)$

We will see that the Schwinger representation of n spins is equivalent to the completely symmetric representation of $SU(2n)$. The algebra of exchange operators generates the subgroup $SU(n)$ while the n sets of spin operators generate the group $[SU(2)]^n$.

We begin by defining a set of matrices related to the spin operators of the Schwinger representation. The spin operators $\mathbf{S}^j$ correspond to a vector of $2n \times 2n$ matrices $\boldsymbol{\mathcal{S}}^j$ which are block diagonal with the Pauli matrices $\boldsymbol{\sigma}$ in the j'th 2×2

block. The other terms in the matrix are zero.

$$\boldsymbol{\mathcal{S}}^j = \frac{1}{2}\begin{pmatrix} 0 & & & & & & \\ & \ddots & & & & & \\ & & 0 & & & & \\ & & & \boldsymbol{\sigma} & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{pmatrix} \tag{3.57}$$

We can see immediately that these matrices have the commutation relations required for the spin operators of the n particles. All the spin matrices commute with those of a different spin while the three matrices of a single spin have the angular momentum commutation relations. These spin matrices generate the group $[SU(2)]^n$, n sets of commuting matrices each generating the angular momentum group $SU(2)$.

As there exists a simple matrix analogue for spin operators in the Schwinger representation of spin we can expect such an analogy to continue for the exchange algebra. The operators $\mathbf{E}^{ij}$ in the Schwinger scheme are related to a vector of $n \times n$ matrices $\boldsymbol{\mathcal{E}}^{ij}$. The nonzero terms in these matrices are where the i'th and j'th columns intersect the i'th and j'th rows.

$$\boldsymbol{\mathcal{E}}^{ij} = \frac{1}{2}\begin{pmatrix} 0 & & & & & & & & & \\ & \ddots & & & & & & & & \\ & & 0 & & & & & & & \\ & & & \boldsymbol{\sigma}_{11} & & & & \boldsymbol{\sigma}_{12} & & \\ & & & & 0 & & & & & \\ & & & & & \ddots & & & & \\ & & & & & & 0 & & & \\ & & & \boldsymbol{\sigma}_{21} & & & & \boldsymbol{\sigma}_{22} & & \\ & & & & & & & & 0 & \\ & & & & & & & & & \ddots & \\ & & & & & & & & & & 0 \end{pmatrix} \tag{3.58}$$

These matrices form a matrix algebra with the same commutation relations as the exchange operators in the Schwinger representation. We can see that the matrices span the space of Hermitian $n \times n$ matrices and so the exchange algebra is the Lie algebra $su(n)$. The matrices defined in (3.58) generate the group of $n \times n$ unitary matrices with determinant one, $SU(n)$. So if we take u to be

$$u = \exp(\sum_{ij} \mathbf{c}^{ij} . \boldsymbol{\mathcal{E}}^{ij}) \tag{3.59}$$

then u is in $SU(n)$. In order to define matrices related to the exchange algebra which have the same commutation relations with the matrices $\mathcal{S}^l_m$ as the operators in the Schwinger representation we take the tensor product of $\mathcal{E}^{ij}_k$ with the 2×2 identity matrix I. The elements u generated by the exchange algebra now permute the 2×2 blocks in $[SU(2)]^n$.

By including the commutators of $\boldsymbol{\mathcal{E}}^{ij} \otimes I$ and $\boldsymbol{\mathcal{S}}^l$ we generate a set of $2n \times 2n$ matrices which span the space of Hermitian $2n \times 2n$ matrices. These matrices define the Lie algebra $su(2n)$ and generate the group $SU(2n)$. We see that the algebra defined by the Schwinger operators is therefore also $su(2n)$ and the states $|\boldsymbol{\mu}\rangle$ are vectors in a representation of $SU(2n)$. The existence of unphysical states where the particles have different spins is a necessary consequence of using the larger algebra of operators which generate the permutations of the spins.

The Schwinger representation of the spins is isomorphic to the symmetrised tensor product of $2ns$ copies of the $2n \times 2n$ defining representation of $SU(2n)$, the generators of which are the matrices we have just defined. To confirm this we can check the highest weight of the Schwinger representation. States in a representation of $SU(2n)$ can be labelled using the eigenvalues of a maximal set of commuting matrices, the Cartan sub-algebra. To construct the Cartan sub-algebra we will select the diagonal generators $\mathcal{E}^{ij}_z$ and $\mathcal{S}^l_z$. As with the Schwinger scheme

$$|\boldsymbol{\mu}\rangle = |e_{12}, e_{13}, \ldots, e_{n-1\,n}, m_1, \ldots, m_n\rangle \tag{3.60}$$

To define a highest weight we fix the order the eigenvalues in the definition of $\boldsymbol{\mu}$ and say that a weight vector of eigenvalues $\boldsymbol{\mu}$ is positive if the first non zero eigenvalue is positive. The highest weight state is then the state $|\boldsymbol{\mu}\rangle$ for which e_{12} is a maximum followed by the other eigenvalues in order.

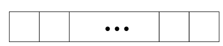
The highest weight state of the Schwinger representation will have all the quanta in the oscillator a_1. This makes e_{12} maximum and m_1 maximum. This highest weight will be $2sn\,|\boldsymbol{\nu}\rangle$ where $|\boldsymbol{\nu}\rangle$ is the highest weight state of the Schwinger representation with just one quanta,

$$|\boldsymbol{\nu}\rangle = |e_{12} = 1/2, \ldots, e_{1n} = 1/2, 0, \ldots, 0, m_1 = 1/2, 0, \ldots, 0\rangle \tag{3.61}$$

This highest weight $\boldsymbol{\nu}$ is also the highest weight of the $2n \times 2n$ defining representation of $SU(2n)$ whose matrix generators we defined. As a basis vector in the defining representation

$$|\boldsymbol{\nu}\rangle = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{3.62}$$

The representation of $SU(2n)$, which has a highest weight that is $2sn$ times the highest weight of the defining representation, is labelled by a Young tableau with a single row of $2sn$ boxes, see section 2.6.4.



This confirms that the Schwinger representation is a symmetrised tensor product of defining representations as those are the symmetry conditions recorded by such a tableau.

3.7 $U(\mathbf{R})$ for n particles

For n particles the operator $U(\mathbf{R})$ which generates the position-dependent spin basis will have the form.

$$U(\mathbf{R}) = \exp(-i \sum_{i<j=1}^{n} \mathbf{c}_{ij}(\mathbf{R}).\mathbf{E}^{ij}) \tag{3.63}$$

The operator must still generate a position dependent basis which is smooth, parallel transports and where

$$|\rho\mathbf{M}(\rho\mathbf{R})\rangle = (-1)^{k\,\mathrm{sgn}(\rho)}|\mathbf{M}(\mathbf{R})\rangle \tag{3.64}$$

which derives from the condition that a single state represents all permutations of particles. We have just seen that the operators $\mathbf{E}^{ij}$ are naturally associated with the group $SU(n)$. In the same way the operator $U(\mathbf{R})$ is connected to an element $u(\mathbf{R})$ of $SU(n)$.

$$u(\mathbf{R}) = \exp(-i \sum_{i<j=1}^{n} \mathbf{c}_{ij}(\mathbf{R}).\boldsymbol{\mathcal{E}}^{ij}) \tag{3.65}$$

$U(\mathbf{R})$ is the Schwinger representation of the element $u(\mathbf{R})$ in the $SU(n)$ subgroup of $SU(2n)$. We can express this as $U(u(\mathbf{R}))$.

We want to find the effect of the permutation condition (3.64) on the elements $u(\mathbf{R})$ of $SU(n)$ used in the construction. To do this we must write the state $|\rho\mathbf{M}\rangle$ in terms of $|\mathbf{M}\rangle$. As all permutations can be written as a product of two cycles we will begin by considering the exchange of the first two particles ρ_{12}. As with the $n = 2$ construction we can use the fixed exchange rotation $\exp(-i\pi E_y^{12})$

$$\begin{aligned} |\rho_{12}\mathbf{M}\rangle &= (-1)^{2s}\exp(-i\pi E_y^{12})|\mathbf{M}\rangle \\ &= (-1)^{2s} U(\exp(-i\pi\mathcal{E}_y^{12}))|\mathbf{M}\rangle \end{aligned} \tag{3.66}$$

The element of $SU(n)$ that produces the exchange is

$$\exp(-i\pi\mathcal{E}_y^{12}) = \begin{pmatrix} 0 & 1 & \\ -1 & 0 & \\ & & I \end{pmatrix} \tag{3.67}$$

where I is the $2n-2$ by $2n-2$ identity matrix. This is an example of a phased permutation matrix

$$\begin{pmatrix} e^{i\phi_1} & & & \\ & e^{i\phi_2} & & \\ & & \ddots & \\ & & & e^{i\phi_n} \end{pmatrix} D(\rho)$$

where $\exp(i\sum\phi_j) = \mathrm{sgn}(\rho)$ and $D(\rho)$ is the $n \times n$ defining representation of the permutation ρ. These phased permutation matrices are a subgroup of $SU(n)$ and the operator U associated with one of them will permute the n spins. For a general permutation ρ

$$|\rho\mathbf{M}\rangle = (-1)^{k\,\mathrm{sgn}(\rho)} U(\{e^{i\phi_1}, \ldots, e^{i\phi_n}\} D(\rho))|\mathbf{M}\rangle \tag{3.68}$$

where the diagonal matrix of phases has been abbreviated $\{e^{i\phi_1}, \ldots, e^{i\phi_n}\}$. From the permutation condition (3.64)

$$U(u^\dagger(\mathbf{R})u(\rho\mathbf{R}))|\rho\mathbf{M}\rangle = (-1)^{k\,\mathrm{sgn}(\rho)}|\mathbf{M}\rangle \tag{3.69}$$

For this to be true for all $|\mathbf{M}\rangle$ the element $u^\dagger(\mathbf{R})u(\rho\mathbf{R})$ of $SU(n)$ must also be a phased permutation matrix. This gives us a condition on the map $u(\mathbf{R})$ from the configuration space to $SU(n)$

$$u(\rho\mathbf{R}) = u(\mathbf{R})\{e^{i\theta_1}, \ldots, e^{i\theta_n}\}D(\rho^{-1}) \tag{3.70}$$

Selecting a map u with this property also defines the operators $U(\mathbf{R})$.

We will now show that $k = 2s$ for any operators $U(\mathbf{R})$ with the required properties. This will involve the exchange of the first two particles ρ_{12} using the fixed exchange rotation $\exp(-i\pi E_y^{12})$. The permutation condition (3.64) can be written as

$$U^\dagger(\mathbf{R})U(\rho_{12}\mathbf{R})\exp(-i\pi E_y^{12})|\mathbf{M}\rangle = (-1)^{k-2s}|\mathbf{M}\rangle \tag{3.71}$$

The related element of $SU(n)$ is then

$$u^\dagger(\mathbf{R})u(\rho_{12}\mathbf{R})\exp(-i\pi\mathcal{E}_y^{12}) =$$

$$\begin{pmatrix} e^{i\theta_1} & & \\ & \ddots & \\ & & e^{i\theta_n} \end{pmatrix} \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & I \end{pmatrix} \begin{pmatrix} 0 & 1 & \\ -1 & 0 & \\ & & I \end{pmatrix} \tag{3.72}$$

This is a diagonal element of $SU(n)$ and so must be generated by the matrices $\mathcal{E}_z^{ij}$.

$$u^\dagger(\mathbf{R})u(\rho_{12}\mathbf{R})\exp(-i\pi\mathcal{E}_y^{12}) = \exp(-i\sum_{ij} c_{ij}\mathcal{E}_z^{ij}) \tag{3.73}$$

Therefore the eigenvalues of the state $|\mathbf{M}\rangle$, for which we have the condition $E_z|\mathbf{M}\rangle = 0$, are unity and consequently in the eigenvalue equation (3.71) k is $2s$.

We have yet to show that there actually exist maps $u(\mathbf{R})$ from the configuration space to $SU(n)$ which have the property (3.70) and produce a smooth position-dependent basis $|\mathbf{M}(\mathbf{R})\rangle$ when n greater than two. This is the statement of the problem reached in [8]. If such a map exists then using the Schwinger representation of spin states

$$|\rho\mathbf{M}(\rho\mathbf{R})\rangle = (-1)^{2s}|\mathbf{M}(\mathbf{R})\rangle \tag{3.74}$$

The spin-statistics connection then follows as in section 3.3.

3.7.1 Maps from configuration space to $SU(n)$

The existence of such a map is an interesting geometrical problem. In [2] [1] and [3] Atiyah constructs a map with the required properties; however there are still important unanswered questions which relate to an alternative more aesthetic construction.

In the simplest but unproved construction $u(\mathbf{R})$ is obtained by orthogonalising an $n \times n$ matrix $w(\mathbf{R})$ where the j'th column of $w(\mathbf{R})$, $\mathbf{w}_j(\mathbf{R})$, is associated with particle j so the permutation condition (3.64) for the columns of the matrix is satisfied.

The direction of the particle i seen from j is t_{ij},

$$t_{ij} = \frac{(\mathbf{r}_i - \mathbf{r}_j)}{|\mathbf{r}_i - \mathbf{r}_j|} \tag{3.75}$$

Each of the $n-1$ directions from the particle j can be described by its complex stereographic coordinates, ζ_i. A unit sphere is centred at $\mathbf{r}_j$ and a line is drawn from the south pole through the point where the line connecting particles j and i intersects the sphere. The line from the south pole intersects the equatorial plane of the sphere and the Cartesian coordinates of the point where it meets the plane provide the real and imaginary parts of ζ_i. The components of $\mathbf{w}_j$, w_{kj}, are the coefficients of z in the polynomial

$$P_j(z) = \prod_{i=1\ldots n-1} (z - \zeta_i) = \sum_{k=1\ldots n} \frac{z^{i-1}}{\sqrt{(k-1)!(n-k)!}} w_{kj} \tag{3.76}$$

To orthogonalise the columns of $w(\mathbf{R})$ the vectors $\mathbf{w}_j(\mathbf{R})$ must be linearly independent for all configurations $\mathbf{R}$. This is surprisingly difficult to prove and to date it hasn't been done although numerical results for $\det w(\mathbf{R})$ are encouraging. It can be proved for $n = 2$ or 3 and in some of the hard cases where it might be expected to fail, for example when all the particles are in a line.

To construct a specific map which has the required properties Atiyah uses a procedure which breaks the translational symmetry of the problem and fixes an origin. The previous construction was independent of the origin used to define the vectors $\mathbf{r}_j$ in $\mathbf{R}$. The construction follows a similar procedure to the more elegant construction described previously. We also define polynomials $P_j(z)$ but now the values of j are distinguished.

(a) If $|\mathbf{r}_j| \geq |\mathbf{r}_i|$ then $t_{ij} = \mathbf{r}_j/|\mathbf{r}_j|$.

(b) If $|\mathbf{r}_j| \leq |\mathbf{r}_i|$ then t_{ij} is the second intersection of the line $(\mathbf{r}_i - \mathbf{r}_j)$ with the sphere of radius $|\mathbf{r}_i|$, see figure 3.1.

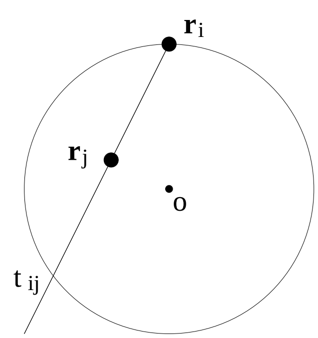


Figure 3.1: The definition of t_{ij} for $|\mathbf{r}_j| \leq |\mathbf{r}_i|$

The complex stereographic coordinates of the directions t_{ij} are again used to define the polynomials. The construction is still compatible with the permutation condition as permuting the coordinates $\mathbf{r}_j$ alters the labels of the points but not the geometry of the configuration which determines the polynomials.

In this construction of the map the polynomials P_j can be shown to be linearly independent. The proof is by induction on n. Take $\mathbf{r}_n$ to be the vector of greatest magnitude, $|\mathbf{r_n}| \geq |\mathbf{r}_j|$ for all j. $Q_1, \ldots Q_{n-1}$ are the polynomials defined by the smaller configuration $(\mathbf{r}_1, \ldots, \mathbf{r}_{n-1})$ of degree $n-2$ which we take to be linearly independent. We choose the complex parameter on the sphere so that $\mathbf{r}_n$ lies at infinity. Then for $j \leq n-1$ the polynomial of the n points is $P_j = Q_j$. While P_n is a polynomial of genuine degree $n-1$ as none of its roots are infinite. Therefore the set of polynomials $P_1, \ldots, P_n$ is still linearly independent. The induction then starts from the trivial case $n = 2$.

With a map from the configuration space of the particles to $SU(n)$ which has the required permutation properties the Schwinger construction of the position-dependent spin basis can proceed. We have already seen that the exchange sign is independent of the particular choice of map and will give the observed spin-statistics relation.

3.7.2 Smoothness and parallel-transport for n particles

The parallel-transport of the position-dependent basis for two particles was demonstrated using the condition

$$E_z^{12}|M\rangle = 0 \tag{3.77}$$

For the Schwinger representation of n spins we still have the condition

$$E_z^{ij}|\mathbf{M}\rangle = 0 \tag{3.78}$$

and the same argument can be applied to show that the spin-basis is parallel transported. We see that this parallel transport condition is also independent of the choice of map $u(\mathbf{R})$ used in the construction.

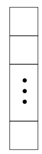
For the position-dependent basis to be a smooth function of $\mathbf{R}$ the map $u(\mathbf{R})$ from configuration space to $SU(n)$ must also be a smooth function of $\mathbf{R}$ up to multiplication on the right by a diagonal matrix. In the first construction the matrix w is smooth up to multiplication of the columns by phase factors and so would generate a smooth position-dependent basis. In the second construction the change between the two regimes as $\mathbf{r}_j$ moves across the sphere of radius $|\mathbf{r}_i|$ is not smooth. The construction is still however continuous and can be made smooth so this is only a technical problem. Given that we know a map $u(\mathbf{R})$ exists which generates a position-dependent basis with the required properties, **3.2.1-3.2.3**, the exchange sign of $|\mathbf{M}(\mathbf{R})\rangle$ is determined by the Schwinger representation of the spins.

3.8 Alternative constructions

Combining the Schwinger representation of spin with the map $u(\mathbf{R})$ found by Atiyah provides constructions of a position dependent spin basis for n particles with all the required properties so that, in this framework, the singlevaluedness of the wavefunction requires that the system obey the spin-statistics theorem. In [9] BR consider alternative constructions with the properties introduced in [8] which don't produce the physically correct spin-statistics relation.

The first alternative construction involves a change in representation of $SU(2n)$. The commutators for the creation and annihilation operators in the Schwinger representation are replaced by anticommutators. Anticommutators imply that $a_j^2 = 0$, with similar relations for the other operators of the harmonic oscillators, so the representation can only have a single quantum in each oscillator. This means that the representation is limited to states of spin-1/2.

In this anticommuting Schwinger representation n spin-1/2 particles are represented by a distribution of n quanta between the oscillators with at most a single quantum in each. The highest weight state of this representation will be the state with the first n oscillators $a_1, b_1, a_2, \ldots$ filled. The weight of this state is the sum of the first n weights of the anti-Schwinger representation with a single quantum. With only a single quantum the anticommutation relations can't affect the available states. So the anticommuting Schwinger representation with a single quantum is isomorphic to the defining representation of $SU(2n)$, which is the commuting Schwinger representation with a single quantum. The sum of the first n maximal weights of the defining representation of $SU(2n)$ is the definition of the n'th fundamental weight and consequently the n spin-1/2 anti-Schwinger representation is isomorphic to the representation of $SU(2n)$ labelled by a Young tableau with a single column of n boxes.



Calculations using the anti-Schwinger scheme show

$$|\rho\mathbf{M}(\rho\mathbf{R})\rangle = +|\mathbf{M}(\mathbf{R})\rangle \tag{3.79}$$

The wavefunctions on this basis will be symmetric under permutations. This is bosonic behaviour for spin 1/2 particles, not the correct spin-statistics relation. This alternative construction is an example of the generalisation we will investigate. In this alternative construction the position-dependent basis has all the properties required in 3.2. We therefore deduce that the spin-statistics connection depends on the representation used for the spin states.

A second alternative construction is for two spin zero particles. In this case the position-dependent spin basis is represented by the unit vector in the direction $\mathbf{r}$

$$|M(\mathbf{r})\rangle = \mathbf{r}/|\mathbf{r}| \tag{3.80}$$

where only one vector, for example $\hat{z}$, corresponds to the spin vector $|0,0\rangle$ in the normal representation of spin. When $-\mathbf{r}$ is substituted for $\mathbf{r}$ this position dependent basis changes sign despite the integer spin. The wrong spin-statistics connection.

Both these alternative constructions of a spin-statistics connection are unsatisfactory. They each apply to a single value of spin and in the second case only to two particles. However they illustrate that the requirements introduced in section 3.2 are insufficient to derive the spin-statistics theorem on their own.

3.9 Parastatistics

One interesting generalisation of the condition **3.2.2** that can be made is to allow a set of states, labelled by an additional quantum number α, to represent both the initial and permuted states. The position-dependent basis is then $|M\,\alpha\,(\mathbf{R})\rangle$ and we can take the fixed spin basis $|\mathbf{M}\,\alpha\rangle$ to be orthonormal.

$$\langle \mathbf{M}'\,\alpha' | \mathbf{M}\,\alpha\rangle = \delta_{\mathbf{M}\mathbf{M}'}\delta_{\alpha\alpha'} \tag{3.81}$$

The permutation condition **3.2.2** is now

$$|\rho\mathbf{M}\,\alpha\,(\rho\mathbf{R})\rangle = \sum_{\beta} c^{\rho}_{\alpha\beta}(\mathbf{R})|\mathbf{M}\,\beta\,(\mathbf{R})\rangle \tag{3.82}$$

and the parallel-transport condition **3.2.3** is

$$\langle \mathbf{M}'\,\alpha'\,(\mathbf{R})|\nabla\,\mathbf{M}\,\alpha\,(\mathbf{R})\rangle = 0 \tag{3.83}$$

Using these conditions we will derive the properties of the coefficients $c^{\rho}_{\alpha\beta}(\mathbf{R})$.

From the orthogonality condition (3.81) we can write

$$\langle \rho\mathbf{M}'\,\alpha'\,(\rho\mathbf{R})|\rho\mathbf{M}\,\alpha\,(\rho\mathbf{R})\rangle = \delta_{\alpha\alpha'}\delta_{\mathbf{M}\mathbf{M}'} \tag{3.84}$$

Using the exchange condition this is

$$\sum_{\beta'\beta} \overline{c}^{\rho}_{\alpha'\beta'}(\mathbf{R})\,c^{\rho}_{\alpha\beta}(\mathbf{R})\,\langle \mathbf{M}'\,\beta'\,(\rho\mathbf{R})|\mathbf{M}\,\beta\,(\rho\mathbf{R})\rangle = \delta_{\alpha\alpha'}\delta_{\mathbf{M}\mathbf{M}'} \tag{3.85}$$

Applying the orthogonality of the states for a second time this reduces to an equation for the coefficients,

$$\sum_{\beta} \overline{c}^{\rho}_{\alpha'\beta}(\mathbf{R})\,c^{\rho}_{\alpha\beta}(\mathbf{R}) = \delta_{\alpha\alpha'} \tag{3.86}$$

If we let $C^{\rho}(\mathbf{R})$ denote a matrix with elements $c^{\rho}_{\alpha\beta}(\mathbf{R})$ then equation (3.86) is equivalent to the matrix equation

$$C^{\rho\dagger}(\mathbf{R})C^{\rho}(\mathbf{R}) = I \tag{3.87}$$

The matrix of coefficients $C^{\rho}(\mathbf{R})$ is unitary.

Using the permutation condition (3.82) we can write

$$\begin{aligned}&\langle\rho\mathbf{M}'\,\alpha'\,(\rho\mathbf{R})|\nabla|\rho\mathbf{M}\,\alpha\,(\rho\mathbf{R})\rangle = \\ &\textstyle\sum_{\beta'\beta}\bar{c}^{\rho}_{\alpha'\beta'}(\mathbf{R})c^{\rho}_{\alpha\beta}(\mathbf{R})\langle\mathbf{M}'\,\beta'\,(\mathbf{R})|\nabla\mathbf{M}\,\beta\,(\mathbf{R})\rangle \\ &+\textstyle\sum_{\beta'\beta}\bar{c}^{\rho}_{\alpha'\beta'}(\mathbf{R})(\nabla c^{\rho}_{\alpha\beta}(\mathbf{R}))\langle\mathbf{M}'\,\beta'\,(\mathbf{R})|\mathbf{M}\,\beta(\mathbf{R})\rangle\end{aligned} \tag{3.88}$$

Applying the parallel transport condition (3.83) on both sides of the equation we find

$$C^{\rho\dagger}(\mathbf{R})\nabla C^{\rho}(\mathbf{R}) = 0 \tag{3.89}$$

As $C^{\rho}(\mathbf{R})$ is unitary this implies

$$\nabla C^{\rho}(\mathbf{R}) = 0 \tag{3.90}$$

The matrix $C^{\rho}(\mathbf{R})$ is a constant and so is independent of $\mathbf{R}$.

Taking the two permutations ρ and σ we will expand the state obtained by applying both permutations to $|\mathbf{M}\,\alpha(\mathbf{R})\rangle$.

$$\begin{aligned}|\rho\sigma\mathbf{M}\,\alpha(\rho\sigma\mathbf{R})\rangle &= C^{\rho}|\sigma\mathbf{M}\,\alpha(\sigma\mathbf{R})\rangle \\ &= C^{\rho}C^{\sigma}|\mathbf{M}\,\alpha(\mathbf{R})\rangle\end{aligned} \tag{3.91}$$

From (3.82) the action of the combined permutation $\rho\sigma$ is

$$|\rho\sigma\mathbf{M}\,\alpha(\rho\sigma\mathbf{R})\rangle = C^{\rho\sigma}|\mathbf{M}\,\alpha(\mathbf{R})\rangle \tag{3.92}$$

We see that the matrices C^{ρ} define a representation of S_n.

$$C^{\rho}C^{\sigma} = C^{\rho\sigma} \tag{3.93}$$

The representation acts on the additional quantum numbers α. Any representation $C(S_n)$ can be decomposed into a direct sum of irreducible representations. Taking a spin state $|\mathbf{M}\gamma\rangle$ in one of these irreducible representations then acting with $U(\mathbf{R})$ we obtain only other vectors in the irreducible representation. Each irreducible subspace is invariant under the action of U. The position-dependent spin basis decomposes into subspaces that transform according to the irreducible representations of S_n in C. In this generalisation of the BR construction it is possible for the position-dependent basis to transform according to any irreducible representation of S_n. Those subspaces transforming according to the higher dimensional representations exhibit parastatistics.

Changing condition **3.2.2** in the construction allows the position-dependent basis to exhibit parastatistics. The same parastatistics will also be present in the wave-functions defined on this basis. By noticing that parastatistics would be consistent with the alternate condition (3.82) we do not in any way change the results for the construction made using the Schwinger representation. In the Schwinger construction parastatistics is not present, the spin states can only transform according to a one dimensional representations of the permutation group which ever version of condition **3.2.2** we impose on the construction.

3.10 The components of the BR construction

To see how the construction can be generalised it will be useful to summarise the essential ingredients that give the BR construction the properties required in section 3.2. This will also demonstrate why the anti-Schwinger construction is viable. Recalling section 3.2, the three requirements on the position-dependent basis are that it is smooth, parallel-transports and under permutations of the positions and spins

$$|\rho\mathbf{M}\,\alpha\,(\rho\mathbf{R})\rangle = \sum_{\beta} c^{\rho}_{\alpha\beta}(\mathbf{R})|\mathbf{M}\,\beta\,(\mathbf{R})\rangle \tag{3.94}$$

This is the more general permutation condition that allows parastatistics. In the previous section by combining (3.94) with the parallel-transport condition we found that states of the position-dependent spin basis must transform under permutations according to an irreducible representation of S_n, equation (3.92).

We will consider the construction of the position dependent basis using the exchange algebra of operators $\mathbf{E}^{ij}$. In order for the operator $U(\mathbf{R})$ to generate a basis that exchanges spins along with positions we saw that the related map $u(\mathbf{R})$ from the configuration space of the n particles to $SU(n)$ must have a permutation property,

$$u(\rho\mathbf{R}) = u(\mathbf{R})\{e^{i\theta_1}, \ldots, e^{i\theta_n}\}D(\rho^{-1}) \tag{3.95}$$

Including parastatistics does not change this requirement. The map should also be smooth up to multiplication by a diagonal matrix if the position-dependent basis is to be smooth. Given such a map we want to be precise about the information used to demonstrate that $|\mathbf{M}(\mathbf{R})\rangle$ has the three required properties.

For the position-dependent basis to be smooth we know already that the map $u(\mathbf{R})$ which defines $U(\mathbf{R})$ must be smooth up to multiplication by a diagonal matrix. Given the permutation condition on the map (3.95) multiplying $U(\rho\mathbf{R})U^\dagger(\mathbf{R})$ by any fixed exchange rotation which permutes the spins produces an operator $\exp(-i\sum_{ij}\alpha(\mathbf{R})_{ij}E_z^{ij})$. We have the condition on the spin states

$$E_z^{ij}|\mathbf{M}\rangle = 0 \tag{3.96}$$

This implies that

$$\exp(-i\sum_{ij}\alpha(\mathbf{R})_{ij}E_z^{ij})|\mathbf{M}\rangle = |\mathbf{M}\rangle \tag{3.97}$$

and there can be no extraneous phase introduced, the position-dependent basis is smooth.

The argument for parallel transport is also based on the spin states $|\mathbf{M}\rangle$ being null states of the z components of the exchange algebra. As long as this is the case the position-dependent basis parallel transports spins.

The permutation condition on the spin states in the position dependent basis is satisfied given any map $u(\mathbf{R})$ with the property (3.95). To demonstrate that the exchange sign is topological, independent of the particular choice of $u(\mathbf{R})$, we also used the property (3.96) of the spin states. We now see that the success of the construction is derived from two basic properties. Firstly the existence of a map $u(\mathbf{R})$ from configuration space to $SU(n)$ with the required permutation property and smooth up to multiplication by a diagonal matrix and secondly that the spin vectors where the n spins are equal are null states of the Cartan sub-algebra of the exchange algebra, (3.96).

We can now see why the anti-Schwinger scheme is also possible. Changing the commutation relations of the creation and annihilation operators does not effect either of these basic properties of the construction. In fact we see that to produce a position-dependent basis with the required properties we have not referred directly to the Schwinger representation of the spin states at all. The Schwinger scheme provides the necessary representation of the operators which form the $su(2n)$ algebra from which the exchange sign is calculated but a different representation of $su(2n)$ could still produce a satisfactory position-dependent basis. A representation $T(SU(2n))$ must simply include states $|\mathbf{M}\rangle$ where the total spins of the n particles are equal and where

$$T(\mathcal{E}_z^{ij})|\mathbf{M}\rangle = 0 \tag{3.98}$$

Any map $u(\mathbf{R})$ used to define a Schwinger construction will then apply equally well to this alternative construction.

3.11 Summary

In this chapter we have seen that in order to generate a position-dependent spin basis which exchanges spins along with positions, the algebra of spin operators can be extended to included an algebra of exchange operators. Spin basis vectors are now vectors in a representation of $SU(2n)$. The permutations of spin with position are generated by a map from the configuration space of the particles to $SU(n)$ the group generated by the exchange algebra. While explicit constructions of this map exist, due to Berry and Robbins for $n = 2, 3$ and Atiyah in the general case, the properties of the position dependent basis under permutations are determined by the representation of spin used and are the same for all maps with the required properties.

The position-dependent basis transforms under permutations of the particles according to a representation of S_n. For the Schwinger spin basis this is the trivial representation for particles of integer spin and the alternating representation for half-integer spin. The transformation properties of the position-dependent basis determine how a wavefunction transforms under permutations. Consequently for the Schwinger scheme wavefunctions on the position dependent spin basis are required to obey the spin-statistics theorem. In order for the position-dependent spin basis to be smooth, parallel-transported and have the correct permutation property, the spin basis vectors must have zero weight with respect to the Cartan sub-algebra of $su(n)$. Replacing the Schwinger representation with another representation of $SU(2n)$ will produce an alternative construction of the position dependent basis. It will have all the required properties but may transform under a different representation of S_n leading to the wrong spin-statistics relation, as occurs with the anti-Schwinger construction. So we are led to the question, *for a general representation of $SU(2n)$ what is the relationship between spin and statistics?*

Chapter 4

Calculation of the exchange sign for the irreducible representations of $SU(4)$

In this chapter we will generalise the Berry-Robbins construction for two spinning particles to find the exchange sign for irreducible representations of $SU(4)$. This is the simplest form of the construction, not only as it has the least particles but the permutation group has only two irreducible representations. The elements of S_2 are I and (12), both form their own class, and the two irreducible representations are the trivial representation and the alternating representation, both one dimensional. In this straightforward case we expect to be able to directly assemble the various spin bases $|M\rangle$ that the representations can act on. The position dependent basis will be constructed in the same way as for the Schwinger representation, by a unitary transformation generated by the exchange algebra. If this subspace transforms under permutations of the particles as the trivial representation of S_2 the particles are behaving as bosons and the wavefunction is symmetric under exchange. Conversely, spin vectors in a subspace transforming according to the alternating representation of S_2 are fermionic, their wavefunctions are antisymmetric under exchange.

4.1 The group $SU(4)$

$SU(4)$ is the group of 4×4 unitary matrices with determinant one. The generators of the group can be defined in terms of the generators of the two spins and of the exchange algebra as in equations (3.57) and (3.58).

$$
\begin{aligned}
\mathcal{S}_{1i} &= \frac{1}{2}\begin{pmatrix} \sigma_i & 0 \\ 0 & 0 \end{pmatrix} & \mathcal{S}_{2i} &= \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & \sigma_i \end{pmatrix} \\
\mathcal{E}_x &= \frac{1}{2\sqrt{2}}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} & \mathcal{E}_y &= \frac{1}{2\sqrt{2}}\begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix} \\
\mathcal{E}_z &= \frac{1}{2\sqrt{2}}\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}
\end{aligned}
\tag{4.1}
$$

σ_i are the Pauli matrices and I is the 2×2 identity matrix. The commutators of these matrices provide the remaining six generators in the algebra. A representation of the group determines a representation of the generators. Let V denote the carrier space of the representation, the spin vectors $|M\rangle$ belong to V.

4.2 The spin subspace

In our discussion of the Schwinger representation in chapter 3 we saw that V is composed of states

$$
|\mu\rangle = (a_1^\dagger)^{n_{a_1}} (a_2^\dagger)^{n_{a_2}} (b_1^\dagger)^{n_{b_1}} (b_2^\dagger)^{n_{b_2}} |0\rangle \tag{4.2}
$$

where the spins of the two particles may be different. Similarly for a general representation only a subspace W of V contains spin states that can be used in the construction. Spin vectors, $|M\rangle$, in W have two properties;

4.2.1 $|M\rangle$ is an eigenvector of $\mathbf{S}_1^2$ and $\mathbf{S}_2^2$ with the same value of total spin s for both particles.

4.2.2 $E_z|M\rangle = 0$; that is, the spin vectors are zero weight states of the exchange algebra. This ensures the position dependent basis is smooth, parallel-transported and that the exchange sign is topological.

These two conditions define the subspace W to which the Berry Robbins construction can be applied. As we showed in chapter 3 the properties of the position dependent basis depend on the commutation relations of the generators, which are a property of the algebra, and the conditions on the spin vectors $|M\rangle$. The problem now breaks down into two parts, finding W for a general representation of $SU(4)$, sections 4.3 and 4.4, and determining the exchange sign for a vector in W, section 4.5 and following.

4.3 Preparing the subspace W

An irreducible representation of $SU(4)$ is labelled by three integers

$$\mathbf{f} = (f_1, f_2, f_3)$$

These are the lengths of the rows of the corresponding Young tableau and we can take $f_1 \geq f_2 \geq f_3$. A vector in an irreducible representation of $SU(4)$ is constructed by taking a tensor product of states of the defining representation of $SU(4)$ and applying a characteristic unit of the symmetric group generated by the tableau $\mathbf{f}$. We will construct vectors in the carrier space of $\mathbf{f}$ which belong to W.

4.3.1 Basis vectors of the tensor product representation

As in the discussion of $su(4)$ in section 2.5.3 the basis vectors of the defining representation are eigenvectors of the Cartan sub-algebra,

$$\mathcal{E}_z = \frac{1}{2\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad \mathcal{S}_{1z} = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.3}$$

$$\mathcal{S}_{2z} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

The eigenvectors

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \mathbf{x}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \mathbf{x}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \mathbf{x}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{4.4}$$

are labelled by weights (e_z, s_{1z}, s_{2z}).

$$\begin{aligned} \boldsymbol{\nu}^1 &= (\tfrac{1}{2\sqrt{2}}, \tfrac{1}{2}, 0) \qquad & \boldsymbol{\nu}^2 &= (\tfrac{1}{2\sqrt{2}}, -\tfrac{1}{2}, 0) \\ \boldsymbol{\nu}^3 &= (-\tfrac{1}{2\sqrt{2}}, 0, \tfrac{1}{2}) \qquad & \boldsymbol{\nu}^4 &= (-\tfrac{1}{2\sqrt{2}}, 0, -\tfrac{1}{2}) \end{aligned} \tag{4.5}$$

The weight $\boldsymbol{\nu}^j$ labels the vector $\mathbf{x}_j$.

A basis vector of the tensor product of defining representations is a tensor product of basis vectors from (4.4)

$$\mathbf{x}' = \mathbf{x}_j \otimes \mathbf{x}_l \otimes \cdots \otimes \mathbf{x}_k \tag{4.6}$$

We will use the notation $N_j(\mathbf{x}')$ for the number of basis vectors $\mathbf{x}_j$ used in $\mathbf{x}'$ and $N(\mathbf{x}')$ for the total number of terms in the tensor product.

4.3.2 Basis vectors of irreducible representations of $SU(4)$

To prepare a vector in the carrier space of $\mathbf{f}$, the vectors $\mathbf{x}_j$ in (4.6) are assigned to boxes of the tableau $\mathbf{f}$. The tensor product is symmetrised with respect to the terms in the rows of the tableau and then antisymmetrised with respect to the columns. This is the application of a primitive characteristic unit of $S_{N(\mathbf{x}')}$ to the tensor product.

For example if we take the representation $(3,1)$ of $SU(4)$ we assemble vectors in V from the tensor product of four basis vectors. Let

$$\mathbf{x}' = \mathbf{x}_1 \otimes \mathbf{x}_4 \otimes \mathbf{x}_2 \otimes \mathbf{x}_4 \tag{4.7}$$

We assign these vectors to the tableau $(3,1)$,

$$\begin{array}{|c|c|c|}\hline 1 & 4 & 2 \\ \hline 4 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}$$

Symmetrising with respect to the rows of the tableau produces

$$\begin{aligned} &\left|\begin{array}{ccc} 1 & 4 & 2 \\ 4 & & \end{array}\right\rangle_S + \left|\begin{array}{ccc} 1 & 2 & 4 \\ 4 & & \end{array}\right\rangle_S + \left|\begin{array}{ccc} 4 & 1 & 2 \\ 4 & & \end{array}\right\rangle_S \\ + &\left|\begin{array}{ccc} 4 & 2 & 1 \\ 4 & & \end{array}\right\rangle_S + \left|\begin{array}{ccc} 2 & 1 & 4 \\ 4 & & \end{array}\right\rangle_S + \left|\begin{array}{ccc} 2 & 4 & 1 \\ 4 & & \end{array}\right\rangle_S \end{aligned}$$

where the ket symbols remind us that the diagrams correspond to tensor products. The kets are labelled with an S to distinguish vectors to which have been symmetrised with respect to the symbols in the rows of the tableau. Finally we antisymmetrise with respect to the vectors in the columns

$$\begin{aligned} &\left|\begin{array}{ccc} 1 & 4 & 2 \\ 4 & & \end{array}\right\rangle_{AS} - \left|\begin{array}{ccc} 4 & 4 & 2 \\ 1 & & \end{array}\right\rangle_{AS} + \left|\begin{array}{ccc} 1 & 2 & 4 \\ 4 & & \end{array}\right\rangle_{AS} - \left|\begin{array}{ccc} 4 & 2 & 4 \\ 1 & & \end{array}\right\rangle_{AS} \\ + &\left|\begin{array}{ccc} 4 & 1 & 2 \\ 4 & & \end{array}\right\rangle_{AS} - \left|\begin{array}{ccc} 4 & 1 & 2 \\ 4 & & \end{array}\right\rangle_{AS} + \left|\begin{array}{ccc} 4 & 2 & 1 \\ 4 & & \end{array}\right\rangle_{AS} - \left|\begin{array}{ccc} 4 & 2 & 1 \\ 4 & & \end{array}\right\rangle_{AS} \\ + &\left|\begin{array}{ccc} 2 & 1 & 4 \\ 4 & & \end{array}\right\rangle_{AS} - \left|\begin{array}{ccc} 4 & 1 & 4 \\ 2 & & \end{array}\right\rangle_{AS} + \left|\begin{array}{ccc} 2 & 4 & 1 \\ 4 & & \end{array}\right\rangle_{AS} - \left|\begin{array}{ccc} 4 & 4 & 1 \\ 2 & & \end{array}\right\rangle_{AS} \end{aligned}$$

The label A denotes that the vector has now been antisymmetrised with respect to the columns. The vectors in the central row cancel leaving

$$\begin{aligned} &\left|\begin{array}{ccc} 1 & 4 & 2 \\ 4 & & \end{array}\right\rangle_{AS} - \left|\begin{array}{ccc} 4 & 4 & 2 \\ 1 & & \end{array}\right\rangle_{AS} + \left|\begin{array}{ccc} 1 & 2 & 4 \\ 4 & & \end{array}\right\rangle_{AS} - \left|\begin{array}{ccc} 4 & 2 & 4 \\ 1 & & \end{array}\right\rangle_{AS} \\ + &\left|\begin{array}{ccc} 2 & 1 & 4 \\ 4 & & \end{array}\right\rangle_{AS} - \left|\begin{array}{ccc} 4 & 1 & 4 \\ 2 & & \end{array}\right\rangle_{AS} + \left|\begin{array}{ccc} 2 & 4 & 1 \\ 4 & & \end{array}\right\rangle_{AS} - \left|\begin{array}{ccc} 4 & 4 & 1 \\ 2 & & \end{array}\right\rangle_{AS} \end{aligned}$$

This is a basis vector of the representation $(3,1)$ of $SU(4)$. Alternatively the vector could be written as a linear combination of tensor products of the $\mathbf{x}_j$'s where each term is a permutation of (4.7). By recording the vector using tableau we avoid specifying the order of the terms in the tensor product. Each box in the Young tableau corresponds to a particular term in the tensor product but as this correspondence is arbitrary it is convenient to be able to suppress it.

4.3.3 Basis vectors of a representation of $SU(2) \times SU(2)$

The $SU(2) \times SU(2)$ subgroup is generated by $\boldsymbol{\mathcal{S}}_1$ and $\boldsymbol{\mathcal{S}}_2$. A representation of $SU(2) \times SU(2)$ is defined by two representations of $SU(2)$ each of which can be labelled by a Young tableau. Vectors in a representation of $SU(2) \times SU(2)$ are the tensor product of vectors of the two representations of $SU(2)$. The basis vectors of the defining representation of $\boldsymbol{\mathcal{S}}_1$ are $\mathbf{x}_1$ and $\mathbf{x}_2$ while $\mathbf{x}_3$ and $\mathbf{x}_4$ are a basis of $\boldsymbol{\mathcal{S}}_2$. To record a vector in a representation of $SU(2) \times SU(2)$ using tableau we assign the vectors $\mathbf{x}_1$ or $\mathbf{x}_2$ to the first tableau and $\mathbf{x}_3$ or $\mathbf{x}_4$ to the second. For example

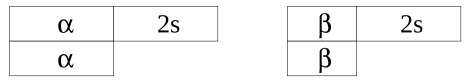
$$\left(\begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 2 \\ \cline{1-1} \end{array} \, , \, \begin{array}{|c|c|} \hline 3 & 4 \\ \hline \end{array} \right)$$

In this case both representations of $SU(2)$ are spin one and the vector has $s_{1z} = s_{2z} = 0$. To construct vectors in an irreducible representation of $SU(2) \times SU(2)$ we apply the $SU(2)$ symmetry conditions for both $SU(2)$ tableau to a vector in the tensor product representation.

4.3.4 $s \otimes s$ multiplets in representations of $SU(4)$

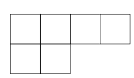
The unitary groups have been applied successfully to problems in particle physics concerning the decomposition of a representation into irreducible components of a subgroup of $U(n)$. In [37] Itzykson and Nauenberg use Young tableau to decompose an irreducible representation of $SU(m+n)$ into representations of the $SU(m) \times SU(n)$ subgroup. This problem is related to finding the subspace W defined previously. Applying their results to $SU(4) \supset SU(2) \times SU(2)$ gives the number of $s_1 \otimes s_2$ multiplets in a representation of $SU(4)$. The multiplets where the spin eigenvalues are both equal form the subspace where the spins are the same, defined in **4.2.1**. This subspace contains W.

Representations of $SU(2) \times SU(2)$ where both components have spin s are labelled by two $SU(2)$ tableau both of which have $2s$ columns of length one.

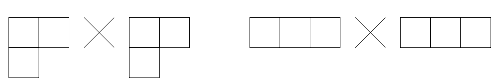


Columns of two boxes don't affect the representation of $SU(2)$ labelled by the tableau, see section 2.6.4. Representations of $SU(4)$ which, when restricted to the $SU(2)\times SU(2)$ subgroup, contain this representation are those whose tableau appear when the two tableau labelling the representation of $SU(2) \times SU(2)$ are multiplied. Multiplying tableau, see section 2.9.1 for the rules, doesn't change the total number of boxes so the number of boxes in the representation of $SU(4)$ is $4s+2\alpha+2\beta$. This is even and so only representations of $SU(4)$ labelled by tableau $\mathbf{f}$ where $|\mathbf{f}|$ is even can contain $s \otimes s$ multiplets. This agrees with the known results for the Schwinger representations which correspond to tableau with a single row of $4s$ boxes, as in 3.6.

We see that when constructing the subspace W we need only consider representations where $|\mathbf{f}|$ is even. Representations with $|\mathbf{f}|$ odd contain no multiplets where the spins of the two particles are equal. If the same tableau $\mathbf{f}$ appears when two pairs of tableau for different values of spin, s and s', are multiplied then $\mathbf{f}$ has both $s' \otimes s'$ and $s \otimes s$ multiplets. This is a common situation, for example the tableau



appears in the product of both the tableau multiplications



Consequently the representation $(4,2)$ of $SU(4)$ contains $s\otimes s$ multiplets for $s = 1/2$ and $s = 3/2$. We can see that a general representation of $SU(4)$ is likely to be more complex than the Schwinger scheme, selecting the representation of $SU(4)$ will not in general fix the value of spin and there may be many multiplets with the same value of spin.

4.3.5 Spin vectors with zero weight with respect to the exchange algebra

Condition **4.2.2** used to define the subspace W requires vectors in W to be eigenvectors of E_z with eigenvalue zero. In the tensor product representation the generator E_z is represented by E'_z,

$$E'_z = \sum_{\text{all permutations}} I \otimes \cdots \otimes I \otimes \mathcal{E}_z \otimes I \otimes \cdots \otimes I \tag{4.8}$$

The sum is over all the possible positions of $\mathcal{E}_z$ in the tensor product. Using (4.5) we see that $\mathbf{x}'$ is an eigenvector of E'_z with eigenvalue

$$e_z = \frac{1}{2\sqrt{2}}(N_1(\mathbf{x}') + N_2(\mathbf{x}') - N_3(\mathbf{x}') - N_4(\mathbf{x}')) \tag{4.9}$$

Similarly $\mathbf{x}'$ is also an eigenvector of S'_{1z} and S'_{2z} with eigenvalues

$$s_{1z} = \frac{1}{2}(N_1(\mathbf{x}') - N_2(\mathbf{x}')) \tag{4.10}$$

$$s_{2z} = \frac{1}{2}(N_3(\mathbf{x}') - N_4(\mathbf{x}')) \tag{4.11}$$

From 4.3.2 a basis vector in an irreducible representation $\mathbf{f}$ of $SU(4)$ is a linear combination of vectors $\mathbf{x}'$ all of which have the same weight (e_z, s_{1z}, s_{2z}). Consequently such a basis vector of an irreducible representation of $SU(4)$ also has weights (4.9), (4.10) and (4.11), where $N_j(\mathbf{x}')$ is the number of vectors $\mathbf{x}_j$ used to prepare the vector in V.

From (4.9) $\mathbf{x}'$ is a null vector of E_z if and only if,

$$N_1(\mathbf{x}') + N_2(\mathbf{x}') = N_3(\mathbf{x}') + N_4(\mathbf{x}') \tag{4.12}$$

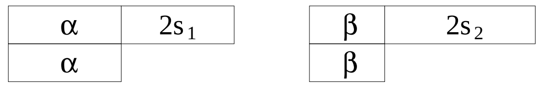
This provides one condition on spin vectors $|M\rangle$ in W.

4.4 $s \otimes s$ multiplets with E_z eigenvalue zero

An irreducible representation $\Gamma_{\mathbf{f}}(SU(4))$ can be restricted to the subgroup $SU(2) \times SU(2)$ generated by the spin operators. This representation $\Gamma_{\mathbf{f}}(SU(2) \times SU(2))$ can then be decomposed into irreducible components,

$$\Gamma_{\mathbf{f}}(SU(2) \times SU(2)) = \bigoplus_{s_1, s_2, e_z} \left(\Gamma_{s_1, s_2, e_z}(SU(2) \times SU(2)) \right) \tag{4.13}$$

An $s_1 \otimes s_2$ multiplet is labelled by a pair of tableau



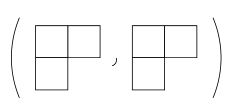
From section 4.3.3 and the eigenvalue equation (4.9) we see that the E_z eigenvalue of such a multiplet is

$$e_z = \frac{1}{2\sqrt{2}}(2s_1 + 2\alpha - 2s_2 - 2\beta) \tag{4.14}$$

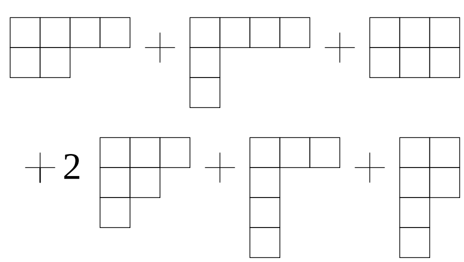
Each $s_1 \otimes s_2$ multiplet has a definite value of e_z. The $s_1 \otimes s_2$ multiplets are distinguished not only by the two spins but also by the eigenvalue of E_z.

The number of $SU(2) \times SU(2)$ multiplets in a representation of $SU(4)$ is the frequency of the tableau $\mathbf{f}$ in the product of the two tableau labelling the multiplet. Combining the condition (4.12) for the E_z eigenvalue of the multiplet to be zero with the condition that $s_1 = s_2$ the multiplet is labelled by two tableau where $\alpha = \beta$. The number of representations $\Gamma_{s,s,0}$ in the decomposition of a representation of $\mathbf{f}$ is the number of tableau $\mathbf{f}$ in the product of the two identical tableau which label the multiplet.

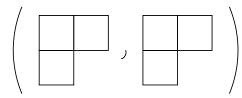
The tableau $(2,1)$ labels a spin-1/2 irreducible representation of $SU(2)$. So vectors in an $s \otimes s$ multiplet with spin-1/2 and E_z-eigenvalue zero can, for example, be labelled by a pair of $(2,1)$ tableau



Multiplying these tableau produces tableau with six boxes.



Each irreducible representations of $SU(4)$ labelled by one of these tableau contains the representation



when restricted to the $SU(2) \times SU(2)$ subgroup. In the example above the representations $(3, 1^3)$ and $(2^2, 1^2)$ of $SU(4)$ are equivalent to those labelled by the partitions (2) and (1^2) respectively. If a tableau appears more than once in the product then this representation of the subgroup contains multiple copies of the representation of $\Gamma_{s,s,0}$ when decomposed into its irreducible components. In our example $(3, 2, 1)$ contains two $s \otimes s$ multiplets with spin-1/2 and $e_z = 0$.

4.4.1 Results for general multiplets

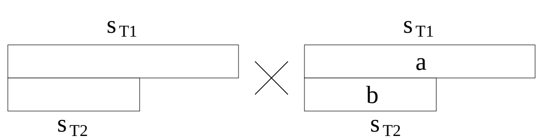
We will consider the general result of multiplying two identical tableau **s** with spin s. As the lengths of the rows of this tableau could easily be confused with the spins of the two particles we will distinguish the two cases with an extra label T for tableau. The partition **s** is then (s_{T1}, s_{T2}) and the spin of the representation of $SU(2)$ labelled by this partition is

$$s = \frac{1}{2}(s_{T1} - s_{T2}) \tag{4.15}$$

As **s** is a tableau we know that

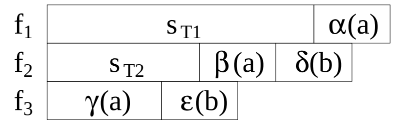
$$s_{T1} \geq s_{T2} \tag{4.16}$$

The multiplication of two such tableau is written



Following the usual rules for multiplying Young tableau we label the boxes in the first row of the second tableau with the symbol "a" and the second row of the tableau "b".

A general result of such a multiplication can be written



Where the a's are in sections of length α, β and γ,

$$\alpha + \beta + \gamma = s_{T1} \tag{4.17}$$

Similarly the b's are in the sections of length δ and ε,

$$\delta + \varepsilon = s_{T2} \tag{4.18}$$

The rows of the resultant tableau have lengths f_1, f_2 and f_3 respectively,

$$2(s_{T1} + s_{T2}) = f_1 + f_2 + f_3 = |\mathbf{f}| \tag{4.19}$$

$|\mathbf{f}|$ is the total number of boxes in the tableau. The general product tableau that we are considering has only three rows. Multiplying two tableau of two rows it is possible, as we saw previously, to produce tableau with columns of four boxes. However tableau with columns of four boxes label a representation of $SU(4)$ equivalent to that labelled by the tableau with the columns of four boxes removed. Considering only the tableau with three rows is sufficient to provide results for all representations of $SU(4)$.

There are several rules used in the multiplication of Young tableau, section 2.9.1. These produce conditions on the lengths of the sections of the resultant tableau. Firstly the result of the multiplication must be a tableau, its rows decreasing in length,

$$f_1 \geq f_2 \geq f_3 \tag{4.20}$$

Then when the boxes labelled with a's are added to the first tableau they can not be put on top of each other. This implies that

$$\beta \leq s_{T1} - s_{T2} \tag{4.21}$$

$$\gamma \leq s_{T2} \tag{4.22}$$

As b's also can't be placed in the same column we obtain a forth condition

$$f_3 \leq s_{T2} + \beta \tag{4.23}$$

Counting right to left and top to bottom the number of a's must always be greater than or equal to the number of b's. This provides two further conditions on the lengths of the sections in our general product tableau.

$$\alpha \geq \delta \tag{4.24}$$

$$\alpha + \beta \geq s_{T2} \tag{4.25}$$

Suppose that given tableau $\mathbf{s}$ and $\mathbf{f}$ there exists a set of variables $\alpha, \beta, \gamma, \delta, \varepsilon$ which satisfy conditions (4.20) to (4.25) and so define one of the tableau of shape $\mathbf{f}$ in the product $\mathbf{s} \times \mathbf{s}$.

$$f_1 = s_{T1} + \alpha \tag{4.26}$$

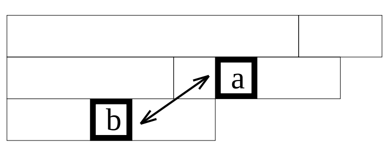
$$f_2 = s_{T2} + \beta + \gamma \tag{4.27}$$

$$f_3 = \delta + \varepsilon \tag{4.28}$$

We will define a procedure that changes the labelling of the boxes in $\mathbf{f}$ without changing its shape. This will be referred to as *procedure* $\mathcal{A}$. $\mathcal{A}$ moves a box labelled b from the third row up to the second and a box a down from the second to the third row.

$$\begin{array}{ccc} \beta & \to & \beta - 1 \\ \delta & \to & \delta + 1 \\ \gamma & \to & \gamma + 1 \\ \varepsilon & \to & \varepsilon - 1 \end{array} \tag{4.29}$$

Schematically this is



We see that not only does $\mathcal{A}$ not change the length of the rows in $\mathbf{f}$ it also doesn't alter the total number of a's or b's in $\mathbf{f}$. Applying $\mathcal{A}$ defines a second labelling of $\mathbf{f}$. To determine if this alternate labelling is a possible result of the tableau multiplication we must check if the new variables $\alpha, \beta', \gamma', \delta', \varepsilon'$ satisfy the conditions (4.20) to (4.25). $\mathcal{A}$ and its inverse are the only such exchanges of boxes which can produce an alternative labelling of the boxes of the tableau which is also a possible result of the multiplication of the two tableau.

To find the total number of tableau of shape $\mathbf{f}$ in the product $\mathbf{s} \times \mathbf{s}$ the idea is to start from the tableau with the maximum number of b's in the third row and then count how many times $\mathcal{A}$ can be applied before the conditions for multiplying tableau are violated. This will give the number of $s \otimes s$ multiplets in the representation $\mathbf{f}$ of $SU(4)$ with $e_z = 0$.

ε is the number of boxes labelled b in the third row of the tableau. First we consider those conditions that restrict the maximum value of ε. ε can not be greater than the length of the third row of $\mathbf{f}$ by definition and it can contain at most all s_{T2} of the b's. In order for ε to be a maximum δ must be a minimum and so β must also be maximum. Condition (4.21) determines the maximum value of β. Collecting these conditions

$$\begin{array}{lcll} \varepsilon & \leq & f_3 & \text{by definition} \\ \varepsilon & \leq & s_{T2} & \text{maximum no. of } b\text{'s} \\ \varepsilon & \leq & s_{T1} + s_{T2} - f_2 & \max \beta \text{ from (4.21)} \end{array} \tag{4.30}$$

The maximum value of ε is equivalently the minimum value of γ, the maximum value of δ or the minimum value of β. There are five conditions which limit the minimum value of ε.

$$\begin{array}{lcll} \varepsilon & \geq & 0 & \text{by definition} \\ \varepsilon & \geq & f_3 - s_{T2} & \text{from (4.22)} \\ \varepsilon & \geq & f_3 - f_2 + s_{T2} & \text{from (4.23)} \\ \varepsilon & \geq & s_{T1} + s_{T2} - f_1 & \text{from (4.24)} \\ \varepsilon & \geq & f_3 + s_{T2} - f_1 & \text{from (4.25)} \end{array} \tag{4.31}$$

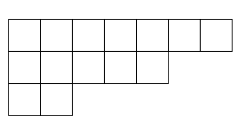
Every ε satisfying conditions (4.30) and (4.31) corresponds to an $s \otimes s$ multiplet. There are three possible maximum values of ε defined by (4.30). The smallest of these maximums is $\varepsilon_{\max}$. Similarly the maximum of the five lower bounds on ε (4.31) is $\varepsilon_{\min}$. The number of ε satisfying the conditions is then $(\varepsilon_{\max} - \varepsilon_{\min} + 1)$. To evaluate this we consider the fifteen combinations that result from combining each of the three upper bounds with each of the five lower bounds. These possible values of $(\varepsilon_{\max} - \varepsilon_{\min} + 1)$ are enumerated in (4.32) below. Four of the combinations are redundant as they are equivalent to other pairs. The variable f_3 has also been eliminated from the results using equation (4.19).

$$\begin{array}{rl} 1) & s_{T2} + 1 \\ 2) & f_1 - f_2 + 1 \\ 3) & f_1 - s_{T1} + 1 \\ 4) & f_1 - 2s_{T2} + 1 \\ 5) & f_2 - s_{T2} + 1 \\ 6) & s_{T1} - s_{T2} + 1 \\ 7) & f_1 + f_2 - 2s_{T1} + 1 \\ 8) & s_{T1} + s_{T2} - f_2 + 1 \\ 9) & f_1 + f_2 - s_{T1} - 2s_{T2} + 1 \\ 10) & f_1 + 2f_2 - 2s_{T1} - 2s_{T2} + 1 \\ 11) & 2s_{T1} + 2s_{T2} - f_1 - f_2 + 1 \end{array} \tag{4.32}$$

To find the number of $s \otimes s$ multiplets in the representation $\mathbf{f}$ we find the minimum of the eleven integers defined in (4.32). If this is negative or zero then there are no such multiplets in $\mathbf{f}$.

It is interesting to note that in fixing the representation of $SU(4)$, even though we do not fix the spin of the multiplets, we do determine whether their spins are integer or half integer. Choosing $\mathbf{f}$ defines the number of boxes $|\mathbf{f}|$ in the tableau and the number of boxes $|\mathbf{s}| = |\mathbf{f}|/2$ in the representations of the spins. Changing the spin $\mathbf{s}$ involves moving boxes from the first row of $\mathbf{s}$ to the second. Moving a single box in this manner changes the number of columns of one box by two which changes the spin by an integer amount. Hence the spins of all multiplets in the representation are either integer or half-integer.

4.4.2 Example: The $s \otimes s$ multiplets of $(7, 5, 2)$ with $e_z = 0$



By choosing a tableau with a relatively large number of boxes and three rows we expect to find several $s \otimes s$ multiplets. As the number of boxes in the tableau is even the representation will contain $s \otimes s$ multiplets with E_z eigenvalue zero. To produce the tableau $(7,5,2)$ the two tableau **s** that we will multiply must each contain seven boxes. This translates to representations of $SU(2)$ with half integer values of spin between 7/2 and 1/2. We can take each value of spin in turn and evaluate the integers (4.32) for these values of f_1, f_2, s_{T1}, s_{T2}.

Starting with spin-1/2, ($s_{T1} = 4, s_{T2} = 3$), evaluating the integers (4.32) we find

$$\begin{aligned} s_{T2} + 1 &= 4 \\ f_1 - f_2 + 1 &= 3 \\ f_1 - s_{T1} + 1 &= 4 \\ f_1 - 2s_{T2} + 1 &= 2 \\ f_2 - s_{T2} + 1 &= 3 \\ s_{T1} - s_{T2} + 1 &= 2 \\ f_1 + f_2 - 2s_{T1} + 1 &= 5 \\ s_{T1} + s_{T2} - f_2 + 1 &= 3 \\ f_1 + f_2 - s_{T1} - 2s_{T2} + 1 &= 3 \\ f_1 + 2f_2 - 2s_{T1} - 2s_{T2} + 1 &= 4 \\ 2s_{T1} + 2s_{T2} - f_1 - f_2 + 1 &= 3 \end{aligned} \tag{4.33}$$

The minimum of these integers is two so there are two spin-1/2 multiplets with $e_z = 0$ in the representation $(7,5,2)$ of $SU(4)$. Continuing this procedure we can fill out a table, figure 4.1, showing the spin of the multiplets available for the construction of the position dependent basis.

Figure 4.1: The $s \otimes s$ multiplets of the representation $(7,5,2)$ of $SU(4)$ with $e_z = 0$

multiplet spin	no. of multiplets
1/2	2
3/2	3
5/2	1
7/2	0

4.5 The exchange sign

So far we have found the subspace of spin vectors W for which the construction of the position-dependent spin basis is defined. As a representation of $SU(4)$ can contain many multiplets with different values of spin we will need to consider how the basis vectors transform under the exchange of the spins in each of these different multiplets.

The position-dependent basis is generated, as in the Schwinger scheme, by a unitary operator $U(\mathbf{r})$. The operator is defined using the same map $u(\mathbf{r})$ from the configuration space to the exchange subgroup $SU(2)$ that was used for the Schwinger representation. In chapter 3 we saw that the properties of the position-dependent basis are independent of the particular form of this map which is chosen, providing that it is smooth up to multiplication by a diagonal matrix of phases and has the desired permutation property

$$u(\rho_{12}\,\mathbf{r}) = u(\mathbf{r}) \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{4.34}$$

For a general representation $\mathbf{f}$ of $SU(4)$

$$U(\mathbf{r}) = \Gamma_{\mathbf{f}}(u(\mathbf{r})) \tag{4.35}$$

As with the Schwinger representation the sign change of vectors in the position-dependent basis generated by $U(\mathbf{r})$ determines the sign of the wavefunction under the exchange of the particles. Consequently evaluating the exchange sign for spin vectors in a multiplet determines the statistics of wavefunctions on the position-dependent basis constructed from it.

4.5.1 Defining a fixed exchange rotation

In the Berry-Robbins construction the exchange sign is independent of the choice of $u(\mathbf{r})$. This topological property of the position-dependent spin basis allows us to evaluate the exchange sign using a fixed exchange rotation. We can use the same exchange rotation as was used to investigate the properties of the Schwinger representation, $\exp(-i\pi\mathcal{E}_y)$.

$$\exp(-i\pi\mathcal{E}_y) = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{4.36}$$

This is an element of the $SU(2)$ exchange subgroup of $SU(4)$. Given any irreducible representation $\Gamma_{\mathbf{f}}$ of $SU(4)$ the element $\exp(-i\pi\mathcal{E}_y)$ defines an exchange rotation which permutes the two spins.

$$\Gamma_{\mathbf{f}}(\exp(-i\pi\mathcal{E}_y))|M\rangle = (-1)^k|\overline{M}\rangle \tag{4.37}$$

where $|M\rangle$ is a vector in the subspace W. For two particles there is no possibility of parastatistics as both irreducible representations of the permutation group S_2 are one-dimensional. To evaluate the exchange sign we want to calculate

$$\langle\overline{M}|\Gamma_{\mathbf{f}}(\exp(-i\pi\mathcal{E}_y))|M\rangle = (-1)^k \tag{4.38}$$

for spin vectors in W.

In the defining representation of $SU(4)$ the exchange rotation $\exp(-i\pi\mathcal{E}_y)$ acts on the basis vectors $\mathbf{x}_j$ transforming them accordingly.

$$\begin{array}{lllllll} \mathbf{x}_1 & \rightarrow & \mathbf{x}_3 & \qquad & \mathbf{x}_2 & \rightarrow & \mathbf{x}_4 \\ \mathbf{x}_3 & \rightarrow & -\mathbf{x}_1 & \qquad & \mathbf{x}_4 & \rightarrow & -\mathbf{x}_2 \end{array}$$

In the tensor product representation the exchange rotation is generated by E'_y,

$$E'_y = \sum_{\text{all permutations}} I \otimes \cdots \otimes I \otimes \mathcal{E}_y \otimes I \otimes \cdots \otimes I \tag{4.39}$$

Using this generator our exchange rotation in the tensor product representation is

$$\exp(-i\pi E'_y) = \exp(-i\pi\mathcal{E}_y) \otimes \exp(-i\pi\mathcal{E}_y) \otimes \cdots \otimes \exp(-i\pi\mathcal{E}_y) \tag{4.40}$$

We can see how this exchange rotation acts on a basis vector $\mathbf{x}'$ in the representation. The vectors $\mathbf{x}_1$ in $\mathbf{x}'$ are replaced with $\mathbf{x}_3$'s and $\mathbf{x}_2$'s with $\mathbf{x}_4$'s and vice versa.

The operator also introduces a sign factor $(-1)^{N_3(\mathbf{x}')+N_4(\mathbf{x}')}$.

Vectors in an irreducible representation $\mathbf{f}$ of $SU(4)$ are generated by applying the symmetry conditions of the tableau $\mathbf{f}$ to a vector $\mathbf{x}'$ in the tensor product representation. For vectors in the subspace W we have already seen that $N_1(\mathbf{x}') + N_2(\mathbf{x}') = N_3(\mathbf{x}') + N_4(\mathbf{x}')$ where the total number of terms in the product is $|\mathbf{f}|$. Operating on a vector with $\Gamma_{\mathbf{f}}(\exp(-i\pi\mathcal{E}_y))$ is equivalent to operating on the vector $\mathbf{x}'$ in the tensor product representation with $\exp(-i\pi E'_y)$ then applying the symmetry conditions of the tableau $\mathbf{f}$ to the result. We see that vectors in the subspace W acquire a sign $(-1)^{|\mathbf{f}|/2}$.

This sign factor does not on its own determine the exchange sign. For example, consider the vector

$$\mathbf{x}_1 \otimes \mathbf{x}_3 - \mathbf{x}_3 \otimes \mathbf{x}_1$$

This is invariant under the operator $\exp(-i\pi E'_y)$; however the sign factor $(-1)^{|\mathbf{f}|/2}$ is -1. The symmetry conditions recorded in the tableau $\mathbf{f}$ also play an important role. We can however separate the two contributions determining the affect of exchanging $\mathbf{x}_1 \leftrightarrow \mathbf{x}_3$ and $\mathbf{x}_2 \leftrightarrow \mathbf{x}_4$ then including the sign factor $(-1)^{|\mathbf{f}|/2}$.

The sign $(-1)^{|\mathbf{f}|/2}$ can be put into a more familiar form using the conditions on $\mathbf{f}$ for it to contain an $s \otimes s$ multiplet with $e_z = 0$. In section 4.4.1 we had condition (4.19)

$$|\mathbf{f}| = 2(s_{T1} + s_{T2}) = 2(2s + 2s_{T2}) \tag{4.41}$$

s_{T2} is an integer but s can be half integer therefore

$$(-1)^{|\mathbf{f}|} = (-1)^{2s} \tag{4.42}$$

This phase factor is reminiscent of the correct spin-statistics connection and forms the first component in our evaluation of the exchange sign for spin vectors in a general representation of $SU(4)$.

4.5.2 Selecting a vector $|M\rangle$

We have simplified the problem by considering only a single exchange rotation $\exp(-i\pi\mathcal{E}_y)$. To simplify it further we will specify a particular vector $|M\rangle$ in each multiplet for which we will determine the exchange sign.

Any spin vector $|M\rangle$ can be written as a series of spin lowering operators S_{1-} and S_{2-} acting on the highest weight state of the representation of $SU(2) \times SU(2)$. The effect of the exchange rotation on these operators can be determined entirely from the commutation relations of the operators. For simplicity if we use the defining representation of $su(4)$ we know that

$$\exp(-i\pi\mathcal{E}_y) = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \tag{4.43}$$

and the spin lowering operators are

$$\mathcal{S}_{1-} = \begin{pmatrix} \mathcal{S}_- & 0 \\ 0 & 0 \end{pmatrix} \qquad \mathcal{S}_{2-} = \begin{pmatrix} 0 & 0 \\ 0 & \mathcal{S}_- \end{pmatrix} \tag{4.44}$$

where all the matrices have been written in 2×2 blocks. Conjugating S_{1-} by $\exp(-i\pi\mathcal{E}_y)$ we obtain the result

$$\exp(-i\pi\mathcal{E}_y)\mathcal{S}_{1-}\{\exp(-i\pi\mathcal{E}_y)\}^{-1} = \mathcal{S}_{2-} \tag{4.45}$$

This is as expected, changing the z component of spin of one of the particles doesn't change the exchange sign. The exchange rotation simply swaps the particle that the spin lowering operator acts on. As our choice of z component of spin for the two particles doesn't effect the exchange sign we will choose the spins of both particles to have maximum z component, s. This spin vector is the highest weight state of the representation of $SU(2) \times SU(2)$ and so is unique. By choosing this state to evaluate (4.38) we know that

$$|\overline{M}\rangle = |M\rangle \tag{4.46}$$

which further simplifies the calculation.

4.5.3 Constructing the highest weight state of a multiplet using Young tableau

In sections 4.3.2 and following we constructed basis vectors in irreducible representations of $SU(4)$ and $SU(2) \times SU(2)$ using Young tableau. To construct the highest weight state of an irreducible representation of $SU(2) \times SU(2)$ we take a pair of tableau labelling the representation of $SU(2) \times SU(2)$. Then basis vectors $\mathbf{x}_1$ of the defining representation are assigned to the first row of the first tableau, $\mathbf{x}_2$ to the second row of the first tableau, $\mathbf{x}_3$ to the first row of the second tableau and $\mathbf{x}_4$ to the second row.

$$\left(\begin{array}{|l|}\hline \qquad 1 \qquad \\ \hline \end{array}\!\!\!\!\!\!\begin{array}{l} \\ \end{array} \begin{array}{|c|}\hline 2 \\ \hline \end{array} , \begin{array}{|l|}\hline \qquad 3 \qquad \\ \hline \end{array} \begin{array}{|c|}\hline 4 \\ \hline \end{array} \right)$$

The tableau is symmetrised with respect to the vectors in the same row, which is trivial in this case, then antisymmetrised with respect to vectors in the same column. The tableau represents the highest weight vector as it contains the maximum number of vectors $\mathbf{x}_1$ and $\mathbf{x}_3$, antisymmetrising columns containing the same symbol would produce zero.

For example the highest weight state of a spin-1/2 multiplet with $e_z = 0$ can be written

$$\left(\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1} \end{array} , \begin{array}{|c|c|}\hline 3 & 3 \\ \hline 4 \\ \cline{1-1} \end{array} \right)$$

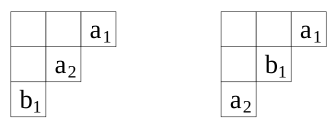
Applying the symmetry conditions of the two $SU(2)$ tableau we have the linear combination of tensor products, written

$$\left| \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1} \end{array} , \begin{array}{|c|c|}\hline 3 & 3 \\ \hline 4 \\ \cline{1-1} \end{array} \right\rangle_{AS} - \left| \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 1 \\ \cline{1-1} \end{array} , \begin{array}{|c|c|}\hline 3 & 3 \\ \hline 4 \\ \cline{1-1} \end{array} \right\rangle_{AS}$$
$$- \left| \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1} \end{array} , \begin{array}{|c|c|}\hline 4 & 3 \\ \hline 3 \\ \cline{1-1} \end{array} \right\rangle_{AS} + \left| \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 1 \\ \cline{1-1} \end{array} , \begin{array}{|c|c|}\hline 4 & 3 \\ \hline 3 \\ \cline{1-1} \end{array} \right\rangle_{AS}$$

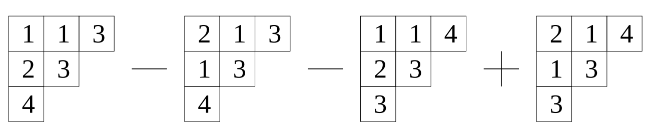
Each box in the pair of tableau corresponds to a position in the tensor product of six basis vectors of the defining representation.

We want to find the highest weight state of an $s \otimes s$ multiplet with $e_z = 0$ in a representation **f** of $SU(4)$. This involves applying the symmetry conditions **f** associated with the multiplet to the highest weight vector of the representation of $SU(2) \times SU(2)$ we defined previously. The symmetry conditions associated with the multiplet are found by multiplying the pairs of tableau in the highest weight state of the representation of $SU(2) \times SU(2)$. We will demonstrate the procedure by continuing the example above discussing more generally what is going on.

In the multiplication of two tableau $(2,1)$ we find the representation $(3,2,1)$ appears twice in the result with the two alternative labelling



Each labelling corresponds to a different multiplet. If we take the first labelling and apply those symmetry conditions to the highest weight state of the spin-1/2 representation of $SU(2) \times SU(2)$ we find



The symmetry conditions of the tableau $(3,2,1)$ labelling the irreducible representation of $SU(4)$ can now be applied to this vector. First the rows in each tableau are symmetrised

$$
\begin{aligned}
&2\left|\begin{array}{ccc}1&1&3\\2&3&\\4&&\end{array}\right\rangle_S - \left|\begin{array}{ccc}2&1&3\\1&3&\\4&&\end{array}\right\rangle_S - 2\left|\begin{array}{ccc}1&1&4\\2&3&\\3&&\end{array}\right\rangle_S + \left|\begin{array}{ccc}2&1&4\\1&3&\\3&&\end{array}\right\rangle_S \\
&+2\left|\begin{array}{ccc}1&1&3\\3&2&\\4&&\end{array}\right\rangle_S - \left|\begin{array}{ccc}2&3&1\\3&1&\\4&&\end{array}\right\rangle_S - 2\left|\begin{array}{ccc}4&1&1\\2&3&\\3&&\end{array}\right\rangle_S + \left|\begin{array}{ccc}2&4&1\\1&3&\\3&&\end{array}\right\rangle_S \\
&+2\left|\begin{array}{ccc}3&1&1\\2&3&\\4&&\end{array}\right\rangle_S - \left|\begin{array}{ccc}1&2&3\\3&1&\\4&&\end{array}\right\rangle_S - 2\left|\begin{array}{ccc}1&4&1\\2&3&\\3&&\end{array}\right\rangle_S + \left|\begin{array}{ccc}4&2&1\\1&3&\\3&&\end{array}\right\rangle_S \\
&+2\left|\begin{array}{ccc}1&3&1\\3&2&\\4&&\end{array}\right\rangle_S - \left|\begin{array}{ccc}1&3&2\\3&1&\\4&&\end{array}\right\rangle_S + \left|\begin{array}{ccc}4&1&2\\1&3&\\3&&\end{array}\right\rangle_S \\
&- \left|\begin{array}{ccc}3&2&1\\1&3&\\4&&\end{array}\right\rangle_S \\
&- \left|\begin{array}{ccc}3&1&2\\1&3&\\4&&\end{array}\right\rangle_S
\end{aligned}
$$

States are multiplied by a factor of two when two different permutations of the symbols in the same row lead to the same labelling of the tableau. To simplify the result all the tableau with two identical symbols in the same column which would vanish when the columns are antisymmetrised have been omitted.

We could now antisymmetrise the tableau with respect to the symbols in the same column however it would produce pages of tableau which can be avoided. If two of the tableau above contain the same symbols in their columns then they will both produce the same set of tableau when antisymmetrised. However the two sets of tableau could differ by a sign. To determine this sign we can compare the two tableau and to see whether an even or odd permutation must be used to rearrange one tableau into the other. For example the two tableau

$$
\begin{array}{ccc}1&1&3\\2&3&\\4&&\end{array} \qquad \begin{array}{ccc}2&1&3\\1&3&\\4&&\end{array}
$$

both produce the same set of tableau when antisymmetrised but with opposite signs as a single exchange of the first two symbols in the first column transforms the second tableau into the first. Using this we can collect tableau that are related by column permutations without antisymmetrising and writing the results out in full.

In our example collecting tableau we are left with

$$3\left|\begin{array}{ccc}1&1&3\\2&3&\\4&&\end{array}\right\rangle_S + 3\left|\begin{array}{ccc}3&1&1\\2&3&\\4&&\end{array}\right\rangle_S + 3\left|\begin{array}{ccc}1&1&3\\3&2&\\4&&\end{array}\right\rangle_S$$
$$+3\left|\begin{array}{ccc}2&4&1\\1&3&\\3&&\end{array}\right\rangle_S + 3\left|\begin{array}{ccc}4&1&2\\1&3&\\3&&\end{array}\right\rangle_S + 3\left|\begin{array}{ccc}2&1&4\\1&3&\\3&&\end{array}\right\rangle_S$$

where the full vector is still found by antisymmetrising the columns. Leaving the vector in this form is sufficient to compare pairs of vectors as we will do later. We have constructed a highest weight vector of the $s\otimes s$ multiplets with spin-1/2 in the representation (321) of $SU(4)$.

With this picture of the procedure the properties of a general construction of the highest weight vector of a multiplet are clearer. A state created by applying first the $(\mathbf{s},\mathbf{s})$ symmetry conditions of the $SU(2)\times SU(2)$ multiplet followed by the symmetry conditions $\mathbf{f}$ of $SU(4)$ is a vector in the representation $\mathbf{f}$ of $SU(4)$ as it is a linear combination of basis vectors of the representation $\mathbf{f}$. It is also a highest weight state of the $SU(2)\times SU(2)$ subgroup as applying a spin raising operator S_{1+} or S_{2+} will produce zero, consider applying the raising operator to the state before the symmetry conditions $\mathbf{f}$ are applied. The multiplication of the two tableau gives a set of linearly independent ways of combining the symmetry conditions.

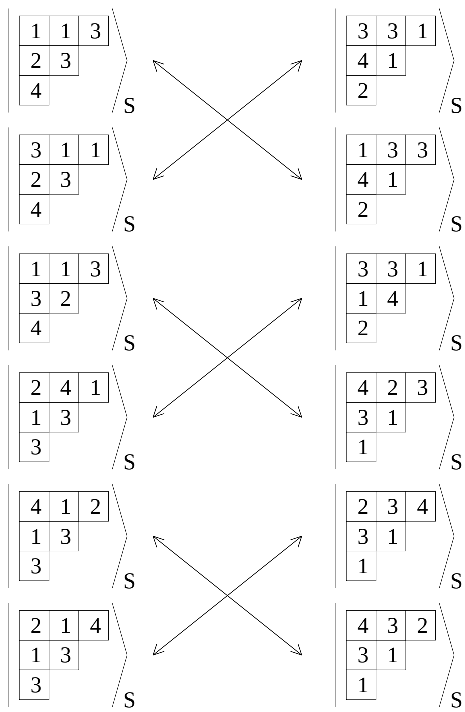
4.5.4 The effect of the symmetry conditions on the exchange sign

The exchange rotation $\exp(-i\pi E'_y)$ changes vectors $\mathbf{x}_1$ to $\mathbf{x}_3$, $\mathbf{x}_2$ to $\mathbf{x}_4$ and vice versa. It also multiplies a vector $|M\rangle$ by the sign factor $(-1)^{2s}$. For the highest weight vector of an $s\otimes s$ multiplet with $e_z=0$ exchanging the spins doesn't change the vector. So from (4.38) the exchange sign is

$$(-1)^k = \langle M|\Gamma_{\mathbf{f}}(\exp(-i\pi\mathcal{E}_y))|M\rangle \tag{4.47}$$

To determine the effect of the symmetry conditions on the exchange sign we will compare the phase of the highest weight state of an appropriate multiplet with the state where the vectors $\mathbf{x}_1$ are replaced with $\mathbf{x}_3$, $\mathbf{x}_2$ with $\mathbf{x}_4$ and vice versa.

To make this clear we can return to our previous example. We will write the tableau recording the highest weight state of the $s \otimes s$ multiplet as a column on the left. The tableau on the right are those where the vectors have been exchanged, $\mathbf{x}_1 \leftrightarrow \mathbf{x}_3$, $\mathbf{x}_2 \leftrightarrow \mathbf{x}_4$.



The arrows connect tableau related by column permutations of the symbols. We can see that the two columns record the same vector as the column permutations that relate the connected tableau are all even. If the permutation required in each case was odd the symmetry conditions would contribute an extra factor of (-1) to the exchange sign.

It is this principle of comparing the highest weight state of the multiplet to the state where the symbols have been exchanged that we will use to evaluate this second contribution to the exchange sign from the symmetry conditions of the Young tableau.

4.5.5 Evaluating the effect of the symmetry conditions for a general highest weight state

A general highest weight state is generated by a general tableau which is itself obtained by filling in the result of the product of two tableau **s** with the symbols 1 to 4. From section 4.4.1 such a general tableau is

$$\begin{array}{l} f_1 \\ f_2 \\ f_3 \end{array} \begin{array}{|c|c|c|}\hline \multicolumn{2}{|c|}{s_{T1}\,(1)} & \alpha\,(3) \\ \hline s_{T2}\,(2) & \beta\,(3) & \delta\,(4) \\ \hline \gamma\,(3) & \varepsilon\,(4) \\ \cline{1-2} \end{array} \tag{4.48}$$

To this tableau the symmetry conditions of the $(\mathbf{s},\mathbf{s})$ tableau are applied permuting 1's with 2's, 3's with 4's. Then the symmetry conditions of the whole tableau $\mathbf{f}$ are applied to form the highest weight vector.

The highest weight vectors generated in this manner span the space of highest weight vectors with $e_z = 0$ for the given spin s. We will show that each vector contains the state labelled by the tableau (4.48) from which the vector is generated. This sounds obvious but will prove useful later. The identity permutation is contained in both sets of symmetry conditions so the question is could the same tableau appear with the opposite sign as the result of a different set of permutations. This is not possible as a sign change introduced in the $(\mathbf{s},\mathbf{s})$ symmetry conditions must be undone with a second antisymmetric permutation to return to the original tableau. The $(\mathbf{s},\mathbf{s})$ symmetry conditions always permute symbols in different rows.

If we consider our previous example applying the $(\mathbf{s},\mathbf{s})$ symmetry conditions

$$\begin{array}{|c|c|c|}\hline 1&1&3\\ \hline 2&3 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array} \longrightarrow \cdots - \left|\begin{array}{|c|c|c|}\hline 2&1&3\\ \hline 1&3 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array}\right\rangle_{AS} \cdots$$

Then applying the symmetry conditions of the tableau $(3,2,1)$ we return to the original tableau with no change in sign.

$$- \left|\begin{array}{|c|c|c|}\hline 2&1&3\\ \hline 1&3 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array}\right\rangle_{AS} \longrightarrow \cdots + \left|\begin{array}{|c|c|c|}\hline 1&1&3\\ \hline 2&3 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array}\right\rangle_{AS} \cdots$$

The highest weight vector contains the state labelled by the tableau used to generate the vector.

The vector generated by (4.48) also includes the state labelled by the exchanged tableau where the symbols 1 and 2 have been swapped with 3 and 4,

$$\left|\begin{array}{lll} \boxed{s_{T1}\,(3)} & \boxed{\alpha\,(1)} & \\ \boxed{s_{T2}\,(4)} & \boxed{\beta\,(1)} & \boxed{\delta\,(2)} \\ \boxed{\gamma\,(1)} & \boxed{\varepsilon\,(2)} & \end{array}\right\rangle_{AS} \tag{4.49}$$

To show that this is a possible result of the application of the two sets of symmetry conditions we must show there exists a set of permutations which applied to (4.48) will produce the tableau (4.49). We can also find the sign of the exchanged tableau in the vector by counting the column permutations used.

Starting from the tableau

$$\begin{array}{lll} \boxed{s_{T1}\,(1)} & \boxed{\alpha\,(3)} & \\ \boxed{s_{T2}\,(2)} & \boxed{\beta\,(3)} & \boxed{\delta\,(4)} \\ \boxed{\gamma\,(3)} & \boxed{\varepsilon\,(4)} & \end{array}$$

we will assume that the identity permutation is chosen from the symmetry conditions of $(\mathbf{s}, \mathbf{s})$. We now split the argument for the symmetry conditions of the tableau $\mathbf{f}$ into two parts.

If the sections β and ε don't overlap ($f_3 \leq s_{T2}$) the permutations necessary to produce (4.49) are straightforward. Starting with the row permutations the α 3's in the first row must be swapped with 1's from the first row, this is a symmetric permutation so there is no change in sign. The δ 4's in the second row are also exchanged symmetrically with 2's in the second row. All the other terms must be exchanged antisymmetrically using column permutations. The β 3's in the second row are swapped with 1's in the top row, a sign change of $(-1)^{\beta}$. Similarly the γ 3's are also swapped antisymmetrically, a sign change of $(-1)^{\gamma}$ and the ε 4's are swapped with 2's in the second row, a sign change of $(-1)^{\varepsilon}$. Combining these contributions the sign of the state (4.49) in the vector generated by the tableau (4.48) is

$$(-1)^{\beta+\gamma+\varepsilon} \tag{4.50}$$

If the sections β and ε overlap the exchange of the symbols is more complex as some symbols must be moved twice. In the overlapping section we start with columns of

$$\begin{array}{|c|}\hline 1\\\hline 3\\\hline 4\\\hline\end{array}$$

The 4 must be swapped with a 2 but there are no 2's in the same row or column. To accomplish the exchange we pair this column with one of the columns of three boxes that is not in the overlapping section. This can always be done as the maximum length of ε is s_{T2} the length of the section containing 2's. To exchange the symbols in this pair of columns we proceed as follows

$$\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\ \begin{array}{c}.....\\.....\\.....\end{array}\ \begin{array}{|c|}\hline 1\\\hline 3\\\hline 4\\\hline\end{array}\ \rightarrow\ \begin{array}{|c|}\hline 1\\\hline 3\\\hline 4\\\hline\end{array}\ \begin{array}{c}.....\\.....\\.....\end{array}\ \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}$$

First the positions of the two columns are exchanged. Swapping symbols in the same row does not introduce a sign change. Then using the antisymmetric column permutations

$$\begin{array}{|c|}\hline 1\\\hline 3\\\hline 4\\\hline\end{array}\ \begin{array}{c}.....\\.....\\.....\end{array}\ \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\ \rightarrow\ \begin{array}{|c|}\hline 3\\\hline 4\\\hline 1\\\hline\end{array}\ \begin{array}{c}.....\\.....\\.....\end{array}\ \begin{array}{|c|}\hline 3\\\hline 1\\\hline 2\\\hline\end{array}$$

The combined permutation of both columns is even so again there is no sign change.

The remaining symbols not involved in these pairs of columns are exchanged as in the first case. As the overlapping region has length $f_3 - s_{T2}$ the sign of the state (4.49) in the vector generated by the tableau (4.48) is

$$(-1)^{\beta+\gamma+\rho-3(f_3-s_{T2})} \tag{4.51}$$

If we consider the alternative vector constructed from the tableau with one less 4 in the third row we can compare the sign of the states labelled by the exchanged tableau in the two cases. To be completely transparent we are comparing the signs of the two different states labelled by the two different exchanged tableau each defined from the tableau used to construct the respective vectors. Earlier we referred to the procedure relating the tableau used to construct two such vectors as procedure $\mathcal{A}$, (4.29), where $\mathcal{A}$ sends

$$\begin{aligned} \beta &\rightarrow \beta - 1 \\ \delta &\rightarrow \delta + 1 \\ \gamma &\rightarrow \gamma + 1 \\ \varepsilon &\rightarrow \varepsilon - 1 \end{aligned}$$

Applying $\mathcal{A}$ adds one to γ and subtracts one from β and ε so

$$(-1)^{(\beta+\gamma+\varepsilon)} \rightarrow (-1)^{(\beta+\gamma+\varepsilon)-1} \tag{4.52}$$

For both cases (4.50) and (4.51) the sign of the vectors labelled by the exchanged tableau alternates between vectors generated by tableau related by $\mathcal{A}$.

Let us proceed for the moment as if the vectors generated by such tableau were eigenvectors of the exchange operation $\exp(-i\pi E_y)$, as we will discuss below in general they are not. Given a vector $|M\rangle$ generated by symmetry conditions $(\mathbf{s}, \mathbf{s})$ and $\mathbf{f}$ we have

$$\exp(-i\pi E_y) \left| \begin{array}{|c|c|c|} \hline s_{T1}\ (1) & & \alpha\,(3) \\ \hline s_{T2}\ (2) & \beta\,(3) & \delta\,(4) \\ \hline \gamma\,(3) & \varepsilon\,(4) & \\ \hline \end{array} \right\rangle = (-1)^{2s} \left| \begin{array}{|c|c|c|} \hline s_{T1}\ (3) & & \alpha\,(1) \\ \hline s_{T2}\ (4) & \beta\,(1) & \delta\,(2) \\ \hline \gamma\,(1) & \varepsilon\,(2) & \\ \hline \end{array} \right\rangle \tag{4.53}$$

where the vector is labelled by the tableau to which the symmetry conditions will be applied to generate the vector. For an eigenvector of the exchange operation our previous results show that either

$$\left| \begin{array}{|c|c|c|} \hline s_{T1}\ (1) & & \alpha\,(3) \\ \hline s_{T2}\ (2) & \beta\,(3) & \delta\,(4) \\ \hline \gamma\,(3) & \varepsilon\,(4) & \\ \hline \end{array} \right\rangle = (-1)^{(\beta+\gamma+\varepsilon)} \left| \begin{array}{|c|c|c|} \hline s_{T1}\ (3) & & \alpha\,(1) \\ \hline s_{T2}\ (4) & \beta\,(1) & \delta\,(2) \\ \hline \gamma\,(1) & \varepsilon\,(2) & \\ \hline \end{array} \right\rangle \tag{4.54}$$

if $f_3 \leq s_{T2}$ or

$$\left| \begin{array}{|c|c|c|} \hline s_{T1}\ (1) & & \alpha\,(3) \\ \hline s_{T2}\ (2) & \beta\,(3) & \delta\,(4) \\ \hline \gamma\,(3) & \varepsilon\,(4) & \\ \hline \end{array} \right\rangle = (-1)^{(\beta+\gamma+\varepsilon)-3(f_3-s_{T2})} \left| \begin{array}{|c|c|c|} \hline s_{T1}\ (3) & & \alpha\,(1) \\ \hline s_{T2}\ (4) & \beta\,(1) & \delta\,(2) \\ \hline \gamma\,(1) & \varepsilon\,(2) & \\ \hline \end{array} \right\rangle \tag{4.55}$$

for $f_3 > s_{T2}$. Substituting these results into (4.53) we can evaluate equation (4.47) to obtain the exchange sign.

$$(-1)^k = \begin{cases} (-1)^{2s+(\beta+\gamma+\delta)} & \text{for } f_3 \leq s_{T2} \\ (-1)^{2s+(\beta+\gamma+\delta)-3(f_3-s_{T2})} & \text{for } f_3 > s_{T2} \end{cases} \tag{4.56}$$

for eigenvectors of the exchange operator.

Using the relationship (4.52) the exchange sign (4.56) alternates between vectors generated by tableau related by $\mathcal{A}$. Consequently

$$\begin{array}{ll} N_e = N_o & N_e + N_o \text{ even} \\ |N_e - N_o| = 1 & N_e + N_o \text{ odd} \end{array} \tag{4.57}$$

where N_e is the number of multiplets with exchange sign of $+1$ and N_o is the number with exchange sign -1. Half of the spin multiplets transform with each exchange sign.

This would amount to a derivation of the exchange sign if the vectors generated by the tableau (4.48) were eigenvectors of the exchange operator, this is not necessarily the case. We can define a set of eigenvectors of the exchange operator that will have the correct exchange signs. However the problem is then to show that these vectors are linearly independent. If they are linearly independent we have found the dimension of the subspaces of W with each exchange sign.

Let us start instead with the vector generated by the exchanged tableau

$$\left| \begin{array}{l} \boxed{\quad s_{T1}\ (3)\quad}\boxed{\alpha\,(1)} \\ \boxed{s_{T2}\ (4)}\boxed{\beta\,(1)}\boxed{\delta\,(2)} \\ \boxed{\gamma\,(1)}\boxed{\varepsilon\,(2)} \end{array} \right\rangle \tag{4.58}$$

then applying the same argument that we used on the vector generated by (4.48) we see that the vector contains a state labelled by the original tableau (4.48) and the sign of this state is the same as the sign we calculated in the reversed situation. If the sign of the vector labelled by the exchanged tableau is ± 1 then we can take the linear combination of vectors generated by the two tableau

$$\left| \begin{array}{l} \boxed{\quad s_{T1}\ (1)\quad}\boxed{\alpha\,(3)} \\ \boxed{s_{T2}\ (2)}\boxed{\beta\,(3)}\boxed{\delta\,(4)} \\ \boxed{\gamma\,(3)}\boxed{\varepsilon\,(4)} \end{array} \right\rangle \pm \left| \begin{array}{l} \boxed{\quad s_{T1}\ (3)\quad}\boxed{\alpha\,(1)} \\ \boxed{s_{T2}\ (4)}\boxed{\beta\,(1)}\boxed{\delta\,(2)} \\ \boxed{\gamma\,(1)}\boxed{\varepsilon\,(2)} \end{array} \right\rangle \tag{4.59}$$

The vector generated by acting on this combination of tableau with both sets of symmetry conditions $(\mathbf{s}, \mathbf{s})$ and $\mathbf{f}$ is not zero as it must include vectors labelled by both the original and exchanged tableau. It is also clearly an eigenvector of $\exp(-i\pi E_y)$

with eigenvalue $(\pm 1)(-1)^{2s}$. If there are N multiplets then this procedure defines N highest weight vectors half with each exchange sign. However these vectors could be linearly dependent.

One way to prove that the exchange signs of the multiplets are determined by the signs calculated previously would be to show that these vectors generated by (4.59) are linearly independent. If there is only a single multiplet with spin-s then there is nothing to prove and we have determined the exchange sign of the multiplet. For two multiplets with spin-s the two highest weight vectors generated by tableau (4.59) will have different exchange signs so they are linearly independent. Consequently all pairs of $s \otimes s$ multiplets with $e_z = 0$ will consist of one with each exchange sign.

Tableau where $f_3 = 0$ can contain at most one multiplet with each spin s, to see this consider applying $\mathcal{A}$ to such a tableau. So we have found the exchange signs of multiplets in a representation of $SU(4)$ labelled by a tableau with two rows. For tableau with three row if $f_3 \leq s_{T2}$ we can show that the vectors generated by (4.59) are linearly independent. Take the vector (4.59) after applying the symmetry conditions $(\mathbf{s}, \mathbf{s})$ and $\mathbf{f}$ it must contain a tensor product labelled by the original tableau which has ε columns

1
2
4

It is impossible for tensor products in the state to be labelled by tableau with more columns of $1, 2, 4$. To see this consider the diagram $(f_3 \leq s_{T2})$

<table>
<tr><td>f_1</td><td colspan="4">s_{T1} (1)</td><td colspan="2">α (3)</td></tr>
<tr><td>f_2</td><td colspan="2">s_{T2} (2)</td><td>β (3)</td><td colspan="2">δ (4)</td></tr>
<tr><td>f_3</td><td>γ (3)</td><td>ε (4)</td></tr>
</table>

Using the $(\mathbf{s}, \mathbf{s})$ symmetry conditions the 4's can only be exchanged with 3's in a higher row which will reduce the number of 4's in the third row. Applying the symmetry conditions $\mathbf{f}$ 4's in the first or second row must undergo a row permutation with a 2 or 1 to increase the number of columns of three boxes containing 4's. This prohibits the column created also containing both a 1 and 2 along with the 4.

The same argument can be applied to the vectors generated by the tableau where 1's and 3's and 2's and 4's have been exchanged to show that again the maximum number of $1, 2, 4$ columns is ε. We can now see how to show the vectors (4.59) are linearly independent. The vector generated by tableau with the maximum number of 4's in the third row, $\varepsilon_{\max}$, is linearly independent of the other vectors as other vectors can not contain as many columns of $1, 2, 4$. The procedure can now be iterated to show that the vector generated by tableau with $\varepsilon = \varepsilon_{\max} - m$ columns of $1, 2, 4$ is independent of the vectors generated by tableau where $\varepsilon = \varepsilon_{\max} - k$ for $k > m$. This proves the vectors (4.59) are linearly independent for $f_3 < s_{T2}$ and consequently the exchange signs for these multiplets obey the relations (4.56) and (4.57).

As yet I have been unable to verify that the vectors (4.59) are linearly independent for tableau with $f_3 > s_{T2}$. For these vectors it is possible to increase the number of columns of $1, 2, 4$ but only to a limited extent. It therefore seems likely that these vectors are linearly independent despite the difficulty in proving it.

4.6 Numerical calculation of the exchange sign

To verify our results for the number of $s \otimes s$ multiplets with $e_z = 0$, (4.32), and the exchange sign of the multiplets, (4.56) and (4.57), we can calculate these properties numerically for the irreducible representations of $SU(4)$ of low dimension. To do this we construct projectors onto a subspace of the representation $\mathbf{f}$. For a diagonalisable matrix A with eigenvalues λ_j the projector onto the subspace with eigenvalue λ_i is P_i.

$$P_i = \prod_{j \neq i} \frac{(A - \lambda_i I)}{(\lambda_i - \lambda_j)} \tag{4.60}$$

We must now consider the distinguishing properties of the subspace we will project onto.

From **4.2.1** the subspace must have equal spin eigenvalues for the two spins, $s_1 = s_2 = s$ and the condition **4.2.2** restricts to a subspace with $e_z = 0$. Of the vectors in W we will project onto a subspace of highest weight vectors of the representation of $SU(2) \times SU(2)$ where the z components of the spins are maximal, $s_{1z} = s_{2z} = s$. This subspace is spanned by eigenvectors of the exchange rotation $\exp(-i\pi E_y)$ with eigenvalues ± 1. By projecting onto the subspace with one of these eigenvalues we can determine the number of multiplets with each exchange sign in the representation of $SU(4)$.

The trace of a product of projection matrices is the dimension of the subspace they project onto. Using this we multiply the matrices defined by the symmetry conditions of a Young tableau, which project onto a subspace with those symmetry conditions, and the projectors onto a subspace where

$$\begin{aligned} e_z &= 0 \\ s_{1z} = s_{2z} &= s \\ s_1^2 = s_2^2 &= s(s+1) \end{aligned} \tag{4.61}$$

Taking the trace of this matrix we find the number of $s \otimes s$ multiplets with e_z eigenvalue zero. This product of projectors is multiplied by the projector of the exchange rotation onto the subspace with exchange sign $+1$. Taking the trace again we find the number of multiplets with this exchange sign. By comparing to the results using the projector of the exchange rotation onto the subspace with exchange sign -1 we can verify that the remaining spin multiplets do have the alternate sign under exchange.

The results of the numerical calculations made by following this scheme using MATLAB are displayed in figures 4.2 and 4.3. The representation of $SU(4)$ is labelled with the appropriate tableau and the spins of the physical multiplets in the representation are given along with their exchange sign.

Before discussing the results it is worth commenting on some of the technical problems that limit such calculations. Although the representations of $SU(4)$ appear simple the size of the matrices in the tensor product representation is $4^{|\mathbf{f}|}$. To

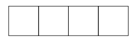
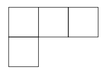
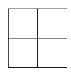
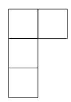
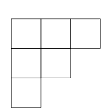
Young Tableau	Multiplet Spin	Exchange Sign
	1	+1
	1	−1
	0 1	+1 +1
	0	−1

Figure 4.2: Numerical results for representations of $SU(4)$ labelled by Young tableau with four boxes

continue the calculation to include the representations labelled by tableau with eight boxes would requires the manipulation of matrices of dimension 65536. There is also an approximately factorial increase in the number of permutation matrices that are required to define the symmetry conditions, the order of the permutation group on $|\mathbf{f}|$ symbols is $|\mathbf{f}|!$ but not all permutations are required for each set of symmetry conditions. The combination of these problems limited calculations to $|\mathbf{f}| = 4$ or 6. To obtain further results the programs could be improved or computing power increased. In particular an efficient algorithm for calculating the permutation matrices is likely to make a significant impact. However even with these improvements the difficulty still grows rapidly with $|\mathbf{f}|$ and obtaining results for tableau of higher dimensions is difficult.

These results for representations of low dimension all agree with the analytic results. Both the number and exchange sign of the multiplets are as predicted. The most interesting case is probably the representation



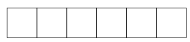
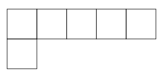
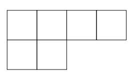
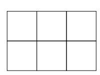
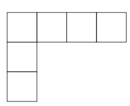
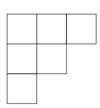
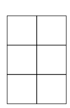
Young Tableau	Multiplet Spin	Exchange Sign
	3/2	−1
	3/2	+1
	1/2 3/2	−1 −1
	1/2 3/2	+1 +1
	1/2	+1
	1/2 1/2	+1 −1
	1/2	−1

Figure 4.3: Numerical results for representations of $SU(4)$ labelled by Young tableau with six boxes

which is the lowest dimensional representation to contain two multiplets with the same spin. We see that the numerics confirm the prediction that the multiplets will have different exchange signs despite belonging to the same representation.

Chapter 5

Character decomposition of $SU(2n)$

The subspace of spin vectors $|\mathbf{M}\rangle$ that can be used to construct the position-dependent basis corresponds to vectors in particular representations of a subgroup $\mathcal{H}$ of $SU(2n)$. The representation of $\mathcal{H}$ from which a vector $|\mathbf{M}\rangle$ is selected determines the spin s of the particles and how the vector transforms under permutations. In order to assemble these physical representations to be used in the Berry-Robbins construction we first consider the group of inner automorphisms of the maximal torus of the exchange rotations, the Weyl group, and determine its action on spin vectors $|\mathbf{M}\rangle$. Both the Weyl group and $\mathcal{H}$ are formulated in terms of the semidirect product which is not necessary but provides a general perspective on the calculation. To find the number of subspaces which correspond to physical representations of the subgroup $\mathcal{H}$ we use the character orthogonality relations.

5.1 Character decompositions

A representation of a group G can be restricted to elements of a subgroup H. This representation of H will in general be reducible. We define χ_H to be the character of an irreducible representation of H and X_G the character of the representation of G. Then the number of irreducible representations corresponding to the character

χ_H in the representation of G is given by

$$N_\chi^X = \frac{1}{\Omega_H} \sum_\lambda \Omega_\lambda \overline{X}_G(h_\lambda) \chi_H(h_\lambda) \tag{5.1}$$

where λ parameterises the classes of H and h_λ is an element of H in the class λ. Ω_λ is the order of the class λ and Ω_H the order of H. This decomposition follows from the character orthogonality relations, which are discussed in more detail in section 2.3.

The character orthogonality described in equation (5.1) assumes that the group is finite. However we will see that the subgroup $\mathcal{H}$ is continuous. For a compact Lie group the sum becomes an integral and the character decomposition looks like

$$N_\chi^X = \frac{1}{\Omega_H} \int \overline{X}_G(h) \chi_H(h)\, \mu(h)\, dh \tag{5.2}$$

where $\mu(h)$ is the Haar measure on H.

We have seen previously that to be used in the Berry-Robbins construction vectors $|\mathbf{M}\rangle$ in the carrier space of the representation of $SU(2n)$ must have certain properties;

5.1.1 $|\mathbf{M}\rangle$ must be an eigenvector of $\mathbf{S}_j^2$ with eigenvalue s for all n spins.

5.1.1 $|\mathbf{M}\rangle$ must be a null state of the z components of the exchange algebra E_z^{ij}.

Only certain representations of $\mathcal{H}$ will correspond to vectors with these properties, we will refer to these as the physical representations of $\mathcal{H}$. In order to use the character orthogonality relations to find the number of physical representations of $\mathcal{H}$ in $SU(2n)$ we must find the classes and characters of $\mathcal{H}$. We will also require the character of the representation of $SU(2n)$ for elements in the subgroup $\mathcal{H}$. With these components we can integrate the product of characters over the subgroup to find the number of physical representations of $\mathcal{H}$ that an irreducible representation of $SU(2n)$ contains.

5.2 The semidirect product

For $SU(2n)$ the action of the Weyl group can be conveniently described using the semidirect product. If we take two groups G and H where G acts on H as a group of automorphisms

$$\Phi_g : H \leftarrow H; h \mapsto h_g \tag{5.3}$$

Then the semidirect product, $G \ltimes H$, is the group with elements (g, h) and the multiplication law

$$(g,h)(g',h') = (gg', hh'_g) \tag{5.4}$$

The identity element of $G \ltimes H$ is (I_G, I_H). From the multiplication law (5.4) we find that

$$(g,h)^{-1} = (g^{-1}, h^{-1}_{g^{-1}}) \tag{5.5}$$

5.2.1 Classes of $G \ltimes H$

All elements in the class of an element (g', h') of $G \ltimes H$ can be found by conjugating (g', h') by an element (g, h) of the group.

$$(g,h)(g',h')(g,h)^{-1} = (gg'g^{-1}, hh'_g h^{-1}_{gg'g^{-1}}) = (g'', h'') \tag{5.6}$$

Elements (g'', h'') in the class of (g', h') are defined by an element g'' of G in the same class of G as g'. The elements h'' that can be obtained by conjugation are restricted to those of the form $hh'_g h^{-1}_{g''}$.

5.2.2 Irreducible representations of $G \ltimes H$

First we determine how G acts on irreducible representations of H. Let $\{(D^\alpha, W_\alpha)\}$ denote a complete set of inequivalent unitary irreducible representations $D^\alpha(H)$ acting on vector spaces W_α. Given (D^α, W_α) let

$$D^\alpha_g(h) = D^\alpha(h_g) \tag{5.7}$$

It is easy to check that $D^\alpha_g(h)$ is a representation of H, it is irreducible and acts on the space W_α. Consequently (D^α_g, W_α) is equivalent to one of the irreducible representations (D^β, W_β) and to denote this instead of β we will use α_g. Given g there exists a unitary transformation

$$\Delta^\alpha(g) : W_\alpha \to W_{\alpha_g}$$

such that

$$D^\alpha(h_g) = \Delta^\alpha(g) D^{\alpha_g}(h) (\Delta^\alpha(g))^{-1} \tag{5.8}$$

Therefore G acts on the set of irreducible representations of H, $\{(D^\alpha, W_\alpha)\}$, by sending α to α_g. The stabiliser of the representation α is a subgroup of G denoted G_α.

$$G_\alpha = \{x \in G : \alpha_x = \alpha\} \tag{5.9}$$

For $x \in G_\alpha$, $\Delta^\alpha(x)$ is a unitary transformation on W_α, and

$$D^\alpha(h_x) = \Delta^\alpha(x) D^\alpha(h) (\Delta^\alpha(x))^{-1} \tag{5.10}$$

We now turn to consider an irreducible representation U of $G \ltimes H$ acting on the carrier space V. We can restrict the representation U to the subgroup H and decompose V into orthogonal subspaces which transform according to an irreducible representation of H labelled by α.

$$V = \bigoplus_\alpha V_\alpha \tag{5.11}$$

If we select a particular subspace V_α which carries the α'th representation of H with multiplicity r then we can choose an orthonormal basis for V_α,

$$|j\mu\rangle$$

where j runs from 1 to d_α the dimension of the representation α and μ runs from 1 to r. In this basis

$$U(h)|j\mu\rangle = D^\alpha_{kj}(h)|k\mu\rangle \tag{5.12}$$

so $U(h)$ is block diagonal in this basis and each block is $D^\alpha(h)$ an irreducible representation of H.

For $x \in G_\alpha$ the representation $U(x)$ leaves V_α invariant.

$$U(x)|j\mu\rangle = \Gamma_{k\nu,j\mu}(x)|k\nu\rangle \tag{5.13}$$

The coefficients $\Gamma_{k\nu,j\mu}(x)$ factorise. To see this we start by using the group multiplication law.

$$U(x)U(h) = U(h_x)U(x) \tag{5.14}$$

Applying this to the state $|j\mu\rangle$ we obtain

$$\Gamma_{l\nu,k\mu}(x)\, D^\alpha_{kj}(h)|l\nu\rangle = D^\alpha_{lk}(h_x)\, \Gamma_{k\nu,j\mu}(x)|l\nu\rangle \tag{5.15}$$

Equating coefficients and using equation (5.10) we get

$$\Gamma_{l\nu,k\mu}(x)\, D^\alpha_{kj}(h) = \Delta^\alpha_{la}\, D^\alpha_{ab}(h)\, (\Delta^\alpha_{bk}(x))^{-1}\, \Gamma_{k\nu,j\mu}(x) \tag{5.16}$$

To simplify this relation we can think of the coefficients $\Gamma_{k\nu,j\mu}$ as defining a d_α-dimensional matrix parameterised by the indices μ and ν,

$$\Gamma_{k\nu,j\mu}(x) = A^{(\nu,\mu)}_{kj}(x) \tag{5.17}$$

so $A^{(\nu,\mu)}(x)$ is a matrix. Writing (5.16) as a matrix equation we have

$$A^{(\nu,\mu)}(x)D^\alpha(h) = \Delta^\alpha(x)D^\alpha(h)(\Delta^\alpha(x))^{-1}A^{(\nu,\mu)}(x) \tag{5.18}$$

Solving this we obtain

$$[(\Delta^\alpha(x))^{-1}A^{(\nu,\mu)}(x)]D^\alpha(h) = D^\alpha(h)[(\Delta^\alpha(x))^{-1}A^{(\nu,\mu)}(x)] \tag{5.19}$$

Schur's lemma implies that the matrix in the square brackets is a multiple of the identity.

$$(\Delta^\alpha(x))^{-1}A^{(\nu,\mu)}(x) = c_{\nu\mu}(x)I \tag{5.20}$$

Returning to the index notation

$$\Gamma_{\mu j,\nu k}(x) = c_{\mu\nu}(x)\, \Delta^\alpha_{jk}(x) \tag{5.21}$$

This is in effect the definition of the tensor product

$$\Gamma = \Delta^\alpha \otimes C \tag{5.22}$$

where C is the matrix with components $c_{\mu\nu}$. This shows that the coefficients Γ factorise.

We are now interested in the properties of Δ and C. As U defines a representation of G we have

$$U(x_1)U(x_2) = U(x_1x_2) \tag{5.23}$$

The matrices Γ also obey this multiplication law,

$$\Gamma(x_1)\Gamma(x_2) = \Gamma(x_1x_2) \tag{5.24}$$

Factorising Γ using equation (5.22) we have the condition

$$\Delta^\alpha(x_1)\Delta^\alpha(x_2) \otimes C(x_1)C(x_2) = \Delta^\alpha(x_1x_2) \otimes C(x_1x_2) \tag{5.25}$$

which implies that

$$\Delta^\alpha(x_1)\Delta^\alpha(x_2) = \gamma(x_1, x_2)\, \Delta^\alpha(x_1x_2) \tag{5.26}$$

$$C(x_1)C(x_2) = \gamma^{-1}(x_1, x_2)C(x_1x_2) \tag{5.27}$$

where $\gamma(x_1, x_2)$ is a phase factor. The equations (5.26) and (5.27) resemble the definitions of a representation of G_α, they are called projective representations. The phase factor $\gamma(x_1, x_2)$ is the factor system of the projective representation. The equations show that both Δ^α and C are projective representations of G_α where the factor systems are the inverses of each other. We will see later that the appearance of these projective representations is not significant. For the representations of $\mathcal{H}$ in which we are interested the projective representations Δ and C will turn out to be representations in the usual sense, $\gamma(x_1x_2) = 1$.

The group G_α is an invariant subgroup of G so we can form the quotient group G/G_α as in section 2.2.2. Elements of this quotient group are cosets gG_α. We can select a set of coset representatives

$$g_1G_\alpha, g_2G_\alpha, \ldots, g_mG_\alpha$$

so that the coset $g_a G_\alpha$ is labelled by a. Any $g \in G$ can be expressed uniquely as a product of a coset representative g_a and an element x in G_α, $g = g_a x$. The product of two coset representatives is

$$g_a g_b = g_{ab} x_{ab}$$

This is determined by the multiplication law of G/G_α. For cosets of a quotient group

$$g_a G_\alpha \, g_b G_\alpha = g_a g_b G_\alpha = g_{ab} G_\alpha \tag{5.28}$$

Using the coset representatives we can define new basis vectors

$$|aj\mu\rangle \equiv U(g_a)|j\mu\rangle \tag{5.29}$$

We will show that the space spanned by the vectors $|aj\mu\rangle$ is invariant under U.

If we take a general element of the representation $U(g,h)$ and act on a basis vector we get

$$\begin{aligned} U(g,h)|bk\nu\rangle &= U(h)U(g)U(g_b)|k\nu\rangle \\ &= U(g)U(g_b)U(h_{g_b^{-1}g^{-1}})|k\nu\rangle \\ &= D^\alpha_{lk}(h_{g_b^{-1}g^{-1}})U(gg_b)|l\nu\rangle \end{aligned} \tag{5.30}$$

Using the coset representatives $gg_b = g_d x$ for a unique x in G_α consequently

$$\begin{aligned} U(g,h)|bk\nu\rangle &= D^\alpha_{lk}(h_{g_b^{-1}g^{-1}})U(g_d)U(x)|l\nu\rangle \\ &= D^\alpha_{lk}(h_{g_b^{-1}g^{-1}})\,\Delta^\alpha_{jl}(x)c_{\mu\nu}(x)U(g_d)|j\mu\rangle \\ &= D^\alpha_{lk}(h_{g_b^{-1}g^{-1}})\,\Delta^\alpha_{jl}(x)c_{\mu\nu}(x)|dj\mu\rangle \end{aligned} \tag{5.31}$$

The vector $U(g,h)|bk\nu\rangle$ is a linear combination of the vectors $|dj\mu\rangle$ so the space spanned by these vectors is invariant under the action of the of U. As U is an irreducible representation the representation C of G_α must be irreducible. The irreducible representations of $G \ltimes H$ are labelled by an irreducible representation α of H and a projective irreducible representation C of G_α with a factor system conjugate to Δ^α.

Equation (5.31) defines the representation U of $G \ltimes H$. To simplify the structure of the equation we can write the basis vectors as tensor products

$$|bk\nu\rangle = |b\rangle \otimes |k\rangle \otimes |\nu\rangle \tag{5.32}$$

Using matrix notation we obtain

$$U(h)\,|b\rangle \otimes |k\rangle \otimes |\nu\rangle \;=\; |b\rangle \otimes D^{\alpha}(h_{g_b^{-1}})|k\rangle \otimes |\nu\rangle \tag{5.33}$$

$$U(g)\,|b\rangle \otimes |k\rangle \otimes |\nu\rangle \;=\; |d\rangle \otimes \Delta^{\alpha}(x)|k\rangle \otimes C(x)|\nu\rangle \tag{5.34}$$

where $gg_b = g_d x$. These equations also define the representation U.

5.3 The Weyl group

We will first define the Weyl group of the exchange permutations $SU(n)$ and then see how this group acts on the spin vectors $|\mathbf{M}\rangle$ which are used to construct the position-dependent spin basis.

5.3.1 Definition of the Weyl group

For a Lie group G the maximal torus T is the group generated by the Cartan subalgebra. The normaliser of G is defined to be

$$\mathrm{Norm}_G(T) = \{x \in G : xtx^{-1} \in T \text{ for all } t \in T\} \tag{5.35}$$

T is clearly an invariant subgroup of $\mathrm{Norm}_G(T)$. We define the *Weyl group* of G as

$$\mathcal{W}_G = \mathrm{Norm}_G(T)/T \tag{5.36}$$

Elements of the quotient group are cosets xT where x is an element $\mathrm{Norm}_G(T)$. There is a natural homomorphism g from $\mathcal{W}_G$ to the automorphisms of T,

$$g(xT)t = xtx^{-1} \tag{5.37}$$

As x is an element of $\mathrm{Norm}_G(T)$ conjugating by x is an automorphism of T. The Weyl group acts on T as automorphisms of T.

5.3.2 The Weyl group for $SU(n)$

For the exchange permutations $SU(n)$ the maximal torus is the group of $n \times n$ diagonal matrices

$$t = \begin{pmatrix} e^{i\theta_1} & & \\ & \ddots & \\ & & e^{i\theta_n} \end{pmatrix} \tag{5.38}$$

where $e^{i\sum\theta_j} = 1$. This is the $n-1$ dimensional torus T^{n-1}.

Applying the definition of the normaliser (5.35) to $SU(n)$ produces conditions on the elements x of $\mathrm{Norm}_{SU(n)}(T^{n-1})$.

$$xt = t'x \tag{5.39}$$

Writing this in component form for the element x_{jk} we get

$$e^{i\theta_k} x_{jk} = e^{i\theta'_j} x_{jk} \tag{5.40}$$

This implies that either $x_{jk} = 0$ or $e^{i\theta_k} = e^{i\theta'_j}$. For a given j the second case can only hold for one k the other entries in the row must all be zero. This implies that x is a phased permutation matrix.

$$x = \begin{pmatrix} e^{i\theta_1} & & \\ & \ddots & \\ & & e^{i\theta_n} \end{pmatrix} D(\rho) \tag{5.41}$$

where $D(\rho)$ is the $n \times n$ defining representation of S_n,

$$D_{jk}(\rho) = \begin{cases} 1 & \text{if } \rho(j) = k \\ 0 & \text{otherwise} \end{cases} \tag{5.42}$$

For the defining representation of the symmetric group,

$$\det D(\rho) = \mathrm{sgn}(\rho) \tag{5.43}$$

so $D(\rho)$ is not an element of $SU(n)$. In order for x to have determinant unity we require $e^{i\sum\theta_j} = \mathrm{sgn}(\rho)$.

We can see that T^{n-1} is indeed a subgroup of these matrices (5.42) obtained by taking ρ to be the identity. The inverse of x is

$$x^{-1} = D^T(\rho) \begin{pmatrix} e^{-i\theta_1} & & \\ & \ddots & \\ & & e^{-i\theta_n} \end{pmatrix} = x^\dagger$$

so these phased permutations x are unitary. The phased permutation matrices (5.41) form the group $\mathrm{Norm}_{SU(n)}(T^{n-1})$. We will call this group of matrices Σ_n to indicate the similarity with S_n. The Weyl group of the exchange angular momentum is $\mathcal{W}_{SU(n)} = \Sigma_n / T^{n-1}$, so $\mathcal{W}_{SU(n)}$ is isomorphic to S_n.

5.3.3 The group Σ_n

We will see later that Σ_n plays an important role in defining $\mathcal{H}$ and in preparation we will investigate the structure of this group more closely. We can parameterise the elements x of Σ_n by the n angles $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_n)$ and the element ρ of S_n. An element x is then defined by $(\rho, \boldsymbol{\theta})$. If we look at the multiplication of two elements x and x' of Σ_n using the factorisation (5.41) we find that

$$(\rho, \boldsymbol{\theta})\,(\rho', \boldsymbol{\theta}') = (\rho\rho', \boldsymbol{\theta} + \rho^{-1}(\boldsymbol{\theta}')) \tag{5.44}$$

This is the multiplication law of semidirect product defined in (5.4)

$$\Sigma_n = S_n \ltimes T^{n-1} \tag{5.45}$$

where S_n acts on T^{n-1} by permuting the angles, $\boldsymbol{\theta} \mapsto \rho^{-1}(\boldsymbol{\theta})$.

5.3.4 Classes of Σ_n

We can now identify the classes of Σ_n. First we note that the inverse of an element x is

$$(\rho, \boldsymbol{\theta})^{-1} = (\rho^{-1}, -\rho(\boldsymbol{\theta})) \tag{5.46}$$

Now if we take an element $(\rho, \boldsymbol{\theta})$ and conjugate with any element $(\sigma, \boldsymbol{\phi})$ we obtain the other elements in the class of $(\rho, \boldsymbol{\theta})$.

$$\begin{aligned}(\rho', \boldsymbol{\theta}') &= (\sigma, \boldsymbol{\phi})(\rho, \boldsymbol{\theta})(\sigma, \boldsymbol{\phi})^{-1} \\ &= (\sigma\rho\sigma^{-1}, \boldsymbol{\phi} + \sigma^{-1}(\boldsymbol{\theta}) - \sigma^{-1}\rho^{-1}\sigma(\boldsymbol{\phi})) \end{aligned} \tag{5.47}$$

The conjugate element ρ' of S_n is in the same class of S_n as ρ. So one label for a classes of Σ_n is a class of S_n.

We must also see how the phases $\boldsymbol{\theta}$ affect the class. If we look at the class of $(I, \boldsymbol{\theta})$ we see that

$$(\rho', \boldsymbol{\theta}') = (I, \sigma^{-1}(\boldsymbol{\theta})) \tag{5.48}$$

σ can be any permutation so all other elements in the class of $(I, \boldsymbol{\theta})$ can be obtained from a permutation of the phases $\boldsymbol{\theta}$. These classes are labelled by unordered sets of n phases $\{\boldsymbol{\theta}\}$.

If we consider the general vector of phases $\boldsymbol{\theta}'$

$$\boldsymbol{\theta}' = \boldsymbol{\phi} + \sigma^{-1}(\boldsymbol{\theta}) - \sigma^{-1}\rho^{-1}\sigma(\boldsymbol{\phi}) \tag{5.49}$$

$\boldsymbol{\phi}$ and σ are arbitrary. Applying the permutation σ to both sides

$$\sigma(\boldsymbol{\theta}') = \sigma(\boldsymbol{\phi}) + \boldsymbol{\theta} - \rho^{-1}\sigma(\boldsymbol{\phi}) \tag{5.50}$$

As $\boldsymbol{\phi}$ is arbitrary let us define a new arbitrary vector of phases, $\boldsymbol{\psi} = \sigma(\boldsymbol{\phi})$.

$$\sigma(\boldsymbol{\theta}') = \boldsymbol{\psi} + \boldsymbol{\theta} - \rho^{-1}(\boldsymbol{\psi}) \tag{5.51}$$

In this general case it is clear that, unlike the classes of the identity element of S_n, the phases $\boldsymbol{\theta}$ are not constants of the conjugation. However for each cycle of ρ^{-1} there is a constant formed from a sum of the phases. Assume ρ^{-1} contains the m cycle $(ij \ldots k)$ then taking the sum of the the $i, j, \ldots, k$'th phases on the right hand side of equation (5.51) we find

$$\psi_i + \theta_i - \psi_j + \psi_j + \theta_j - \cdots + \psi_k + \theta_k - \psi_i = \sum_{l \,\mathrm{in}\, (ij \ldots k)} \theta_l$$

We will define the sum of the phases in a cycle to be

$$\theta_{ij\ldots k} = \sum_{l \text{ in } (ij\ldots k)} \theta_l \tag{5.52}$$

Applying any permutation σ^{-1} the vector of phases $\boldsymbol{\theta}'$ will still contain m phases whose sum is $\theta_{ij\ldots k}$. Setting the values of these constants determines the class of $(\rho, \boldsymbol{\theta})$ for a given ρ. A class of Σ_n is determined by r angles where r is the number of cycles in ρ.

The classes of S_n are labelled by a partition λ, where $|\lambda| = n$. In a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ the integers λ_j correspond to the lengths of the cycles of elements of S_n in the class λ. We label the classes of Σ_n with a partition λ and the vector of angles $\boldsymbol{\theta}_\lambda = (\theta_{\lambda_1}, \ldots, \theta_{\lambda_r})$. The condition on the phases, $e^{i\sum\theta_j} = \mathrm{sgn}(\rho)$, transfers to a condition on the parameters θ_{λ_j}

$$e^{i\sum_{j=1}^{r}\theta_{\lambda_j}} = \mathrm{sgn}(\rho) \tag{5.53}$$

$\mathrm{sgn}(\rho)$ is constant for any permutation ρ in the class λ as the number and length of the cycles is the same for all elements in λ. Using the condition (5.53) the number of parameters θ_{λ_j} can be reduced by one. A class of Σ_n is therefore labelled by a partition λ of n into r integers and $r-1$ angles θ_{λ_j} for each partition.

5.3.5 Representations of Σ_n

Σ_n is the semidirect product $S_n \ltimes T^{n-1}$ so, applying the results of section 5.2.2, the irreducible representations of Σ_n are labelled by irreducible representations of T^{n-1} and projective irreducible representations of the stabiliser of S_n with respect to the representation of T^{n-1}.

T^{n-1} is the group of diagonal matrices defined in (5.38). The group is Abelian so the irreducible representations are one dimensional. The irreducible representations Q of T^1 are labelled by an integer m_1 where

$$Q^{m_1}(\theta_1) = e^{im_1\theta_1} \tag{5.54}$$

An irreducible representation of T^{n-1} is formed from a product of these monomial representations.

$$Q^{\mathbf{m}}(\boldsymbol{\theta}) = e^{i\sum m_j\theta_j} \tag{5.55}$$

The irreducible representation Q is labelled by the vector of n integers, $\mathbf{m} = (m_1, \ldots, m_n)$. As Q is a representation of T^{n-1} we expect one of these integers to be redundant. For the group T^{n-1} we have the condition, $e^{i\sum\theta_j} = 1$. If we define a vector of integers $\mathbf{m}'$ from $\mathbf{m}$ so that

$$\mathbf{m}' = (m_1 - m_n, m_2 - m_n, \ldots, m_{n-1} - m_n, 0) \tag{5.56}$$

then we see that the representation $Q^{\mathbf{m}}$ is equivalent to $Q^{\mathbf{m}'}$. The vector $\mathbf{m}$ labels irreducible representations of T^{n-1}.

In the semidirect product the group S_n of automorphisms of T^{n-1} defines a map between irreducible representations of T^{n-1}. For ρ in S_n

$$Q^{\mathbf{m}}(\rho^{-1}(\boldsymbol{\theta})) = e^{i\sum m_j\theta_{\rho^{-1}(j)}} = Q^{\rho(\mathbf{m})}(\boldsymbol{\theta}) \tag{5.57}$$

The stabiliser of S_n with respect to the representation $\mathbf{m}$ is defined as the group

$$S_{n,\mathbf{m}} = \{\rho \in S_n : Q^{\rho(\mathbf{m})} = Q^{\mathbf{m}}\} \tag{5.58}$$

This is the subgroup of permutations between equal integers in $\mathbf{m}$. If we divide the integers m_j into sets $\{i, j, \ldots, k\}$ where $m_i = m_j = \cdots = m_k$ then $S_{n,\mathbf{m}}$ is the direct product of the symmetric groups on these sets of symbols. A representation of this direct product is defined by choosing a representation for each of the subgroups of permutations amongst the sets $\{i, j, \ldots, k\}$. Representations of Σ_n defined using projective representations of the stabiliser are not considered as they will not be needed later.

To make this clear let us take the representation of T^{n-1} with $n = 6$ labelled by the vector of integers

$$\mathbf{m} = (3, 3, 3, 1, 4, 4)$$

Then the product of the symmetric groups S_3 of permutations of $\{1, 2, 3\}$ and S_2 of $\{5, 6\}$ form the group $S_{n,\mathbf{m}}$. An irreducible representation is labelled by two partitions, λ of three and η of two.

A representation of Σ_n is labelled by a representation $\mathbf{m}$ of T^{n-1} and r partitions λ_j one for each set of equal integers. $|\lambda_j|$ is the number of equal integers in one set. Using the results in section 5.2.2 we can construct an irreducible representation for any choice of $\mathbf{m}$ and $\lambda_1 \ldots \lambda_r$.

5.3.6 The action of the Weyl group on spin vectors $|\mathbf{M}\rangle$

We will now consider the properties that make the Weyl group significant in the construction of the position dependent spin basis. A representation $\Gamma(SU(2n))$ acts on a complex vector space V. We have seen that spin vectors $|\mathbf{M}\rangle$ used in the construction must be in a subspace W of V. From **5.1.2** vectors in W have zero weight with respect to the exchange angular momentum $su(n)$. If we take x to be an automorphism of the maximal torus T^{n-1} of $SU(n)$ the representation $\Gamma(SU(2n))$ can be restricted to elements x of Σ_n. For $|\mathbf{M}\rangle$ in W

$$\Gamma(x)|\mathbf{M}\rangle = \Gamma(xt)|\mathbf{M}\rangle \tag{5.59}$$

where t is any element of T^{n-1}. This comes from the relation for zero weight vectors, $\Gamma(t)|\mathbf{M}\rangle = |\mathbf{M}\rangle$. As a consequence the representation $\Gamma(xt)$ acting on the subspace W is the same for any element of the coset xT and the representation $\Gamma(\Sigma_n)$ is also a representation of the Weyl group Σ_n/T_{n-1} of $SU(n)$.

We see that on the spin subspace W restricting the representation Γ to the automorphisms of $SU(n)$ is equivalent to defining a representation of $\mathcal{W}_{SU(n)}$. We now want to investigate the properties of the Weyl group. An element of $\mathcal{W}_{SU(n)}$ is a coset xT^{n-1}. Using the decomposition (5.41) an element of the coset can be written

$$\begin{pmatrix} e^{i\theta_1} & & \\ & \ddots & \\ & & e^{i\theta_n} \end{pmatrix} D(\rho) \begin{pmatrix} e^{i\alpha_1} & & \\ & \ddots & \\ & & e^{i\alpha_n} \end{pmatrix}$$

$$= \begin{pmatrix} e^{i(\theta_1+\alpha_{\rho^{-1}(1)})} & & \\ & \ddots & \\ & & e^{i(\theta_n+\alpha_{\rho^{-1}(n)})} \end{pmatrix} D(\rho)$$

The only condition on the phases α is that $e^{i\sum\alpha_j} = 1$ so the coset consists of all elements x of Σ_n which are constructed from the same element ρ of S_n. As Σ_n is the semidirect product of S_n and T_{n-1} the multiplication law for the cosets is just that of S_n. It follows that the Weyl group, $\mathcal{W}_{SU(n)}$, is isomorphic to the permutation group S_n.

This is just the structure that we require in order to construct the position dependent basis. In the group $SU(2n)$ we need a subgroup which permutes the n spins, Σ_n. However if the spin states $|\mathbf{M}\rangle$ are to transform according to an irreducible representation of S_n then a representation of Σ_n should also provide a representation of S_n in the spin subspace W. As any representation of Σ_n on W descends to a representation of $\mathcal{W}_{SU(n)}$ spin vectors transform according to a representation of S_n.

5.4 The physical subgroup $\mathcal{H}$

5.4.1 The n spin subgroup

The exchange permutations, which produce $\mathcal{W}_{SU(n)}$, are not the only significant subgroup used to define the subspace of spins W. From **5.1.1** and **5.1.2** spin vectors $|\mathbf{M}\rangle$ in W have zero weight with respect to $su(n)$ and identical spins s with respect to the n spin subgroup $[SU(2)]^n$. Matrices U in $[SU(2)]^n$ are constructed from n

2×2 matrices u_j in $SU(2)$,

$$U = \begin{pmatrix} u_1 & & \\ & \ddots & \\ & & u_n \end{pmatrix} \tag{5.60}$$

These matrices clearly form a subgroup of $SU(2n)$. Irreducible representations of $[SU(2)]^n$ are constructed by taking the tensor product of n irreducible representations of $SU(2)$

$$R(U) = R^1(u_1) \otimes R^2(u_2) \otimes \cdots \otimes R^n(u_n) \tag{5.61}$$

where $R^j(u)$ is an irreducible representation of $SU(2)$. An irreducible representation $R([SU(2)]^n)$ where the n spins are the same is a tensor product of n identical representations of $SU(2)$, $R^1(SU(2)) = \cdots = R^n(SU(2))$.

If we restrict a representation of $SU(2n)$ to the $[SU(2)]^n$ subgroup we can decompose this reducible representation of $[SU(2)]^n$ into irreducible components and find the number of these representations where the n spins are identical. The spin vectors $|\mathbf{M}\rangle$ used in the construction must be chosen from the subspaces spanned by these representations.

5.4.2 Definition of $\mathcal{H}$

We want to restrict a representation of $SU(2n)$ to a subgroup generated by the permutation operations Σ_n and the spin subgroup $[SU(2)]^n$. We will call this the subgroup of physical transformations of the spin vectors, $\mathcal{H}$. The vectors $|\mathbf{M}\rangle$ used in the construction of the transported spin basis will belong to particular irreducible representations of this subgroup.

Given a vector $|\mathbf{M}\rangle$ in the subspace W applying one of these transformations will produce another vector in W with the same spin s. To form elements in $\mathcal{H}$ we take the product of an element of $[SU(2)]^n$ with an element of Σ_n.

$$\begin{pmatrix} u_1 & & \\ & \ddots & \\ & & u_n \end{pmatrix} \begin{pmatrix} e^{i\theta_1} I & & \\ & \ddots & \\ & & e^{i\theta_n} I \end{pmatrix} D(\rho) \otimes I$$
$$= \begin{pmatrix} e^{i\theta_1} u_1 & & \\ & \ddots & \\ & & e^{i\theta_n} u_n \end{pmatrix} D(\rho) \otimes I \tag{5.62}$$

The elements of Σ_n defined in equation (5.41) are made into a subgroup of $SU(2n)$ by taking the tensor product with the 2×2 identity matrix as with the generators of the exchange angular momentum in section 3.6. Elements of $\mathcal{H}$ are parameterised by $\mathbf{u} = (u_1, \ldots, u_n)$, $\boldsymbol{\theta}$ and ρ. As with Σ_n we can write down the multiplication law for elements of $\mathcal{H}$ constructed using equation (5.62).

$$(\rho, \boldsymbol{\theta}, \mathbf{u})\,(\rho', \boldsymbol{\theta}', \mathbf{u}') = (\rho\rho', \boldsymbol{\theta} + \rho^{-1}(\boldsymbol{\theta}'), \mathbf{u}.\rho^{-1}(\mathbf{u}')) \tag{5.63}$$

This is also the multiplication law of a semidirect product,

$$\mathcal{H} = \Sigma_n \ltimes [SU(2)]^n \tag{5.64}$$

where Σ_n acts on $[SU(2)]^n$ by permuting the elements of $SU(2)$, $\mathbf{u} \mapsto \rho^{-1}(\mathbf{u})$ for $(\rho, \boldsymbol{\theta}) \in \Sigma_n$.

5.4.3 Classes of $\mathcal{H}$

The classes of Σ_n are labelled by partitions λ of n into r parts and a vector of angles $\boldsymbol{\theta}_\lambda$ of length r for each partition λ. In section 5.2.1 we noted that classes of a semidirect product are labelled firstly by classes of the automorphism group. The automorphisms of $[SU(2)]^n$ are provided by Σ_n so λ and $\boldsymbol{\theta}_\lambda$ will also distinguish classes of $\mathcal{H}$.

If we conjugate an element $(\rho, \boldsymbol{\theta}, \mathbf{u})$ of $\mathcal{H}$ with all elements $(\sigma, \boldsymbol{\phi}, \mathbf{v})$ we obtain the class of $(\rho, \boldsymbol{\theta}, \mathbf{u})$.

$$
\begin{aligned}
(\rho', \boldsymbol{\theta}', \mathbf{u}') &= (\sigma, \boldsymbol{\phi}, \mathbf{v})\,(\rho, \boldsymbol{\theta}, \mathbf{u})\,(\sigma, \boldsymbol{\phi}, \mathbf{v})^{-1} \\
= (\sigma\rho\sigma^{-1}, \boldsymbol{\phi} + \sigma^{-1}(\boldsymbol{\theta}) &- \sigma^{-1}\rho^{-1}\sigma(\boldsymbol{\phi}), \mathbf{v}.\sigma^{-1}(\mathbf{u}).\sigma^{-1}\rho^{-1}\sigma(\mathbf{u}))
\end{aligned} \tag{5.65}
$$

We are interested in the extent to which $\mathbf{u}'$ is determined by $\mathbf{u}$. To investigate this we will follow the procedure used to determine the classes of Σ_n. If we apply the permutation σ to $\mathbf{u}'$ and define a new arbitrary element of $[SU(2)]^n$ from $\mathbf{v}$

$$
\mathbf{w} = \sigma(\mathbf{u}) \tag{5.66}
$$

Then from (5.65) we find the relation

$$
\sigma(\mathbf{u}') = \mathbf{w}.\mathbf{u}.\rho^{-1}(\mathbf{w}^{-1}) \tag{5.67}
$$

If we consider the element u_1 of $SU(2)$ it is clear that, provided 1 is not fixed by ρ^{-1}, we can obtain any element of $SU(2)$ as the first term in $\sigma(\mathbf{u}')$. If the new first element is to be w_1 then let the element $w_{\rho(1)}$ be u_1. The class of individual elements u_j in $SU(2)$ is not in general maintained by conjugation.

Let σ^{-1} contain the m cycle $(ijk\ldots l)$ then the product of the elements of $\sigma(\mathbf{u}')$ in this cycle is

$$
\begin{aligned}
u'_i u'_j \ldots u'_l &= (w_i u_i w_j^{-1})(w_j u_j w_k^{-1}) \ldots (w_l u_l w_i^{-1}) \\
&= w_i (u_i u_j \ldots u_l) w_i^{-1}
\end{aligned} \tag{5.68}
$$

The result is in the same class of $SU(2)$ as the element $u_i u_j \ldots u_k$. To obtain $\mathbf{u}'$ we apply the permutation σ^{-1}. This changes the order of the elements of $SU(2)$ however there will still be m terms whose product is in the same class of $SU(2)$ as $u_i u_j \ldots u_k$. The class of $SU(2)$ of the product of the elements of $\mathbf{u}$ in the same cycle of σ^{-1} is a constant of conjugation.

The class of an element u of $SU(2)$ is labelled by a complex number ϵ of modulus unity where the two eigenvalues of u are ϵ and $\overline{\epsilon}$, see section 2.7. To distinguish the classes of $\mathcal{H}$ we require one eigenvalue ϵ for each cycle in ρ^{-1}. If the class of ρ is labelled by a partition λ of n into r parts the classes of $\mathcal{H}$ for this choice of σ^{-1} are distinguished by a vector of eigenvalues of $SU(2)$, $\boldsymbol{\epsilon}_\lambda = (\epsilon_1, \ldots, \epsilon_r)$.

As $\mathcal{H}$ is a semidirect product the classes of $\mathcal{H}$ are labelled by classes of Σ_n and the constants of conjugation $\boldsymbol{\epsilon}$. A class of $\mathcal{H}$ is labelled by λ a partition of n into r parts and for each λ a vector of angles, $\boldsymbol{\theta}_\lambda$, and a vector of eigenvalues of $SU(2)$, $\boldsymbol{\epsilon}_\lambda$, both of length r.

5.4.4 The volume element of $\mathcal{H}$

The group $\mathcal{H}$ has an unusual structure, it consists of a finite number of continuous parts one for each element of S_n. To sum a product of characters over the group $\mathcal{H}$ involves a sum over the elements of S_n and an integral over the continuous parameters of the classes for each element of S_n. For two elements ρ and σ in the same class of S_n we have found that the continuous parameters labelling their classes are the same. Therefore the sum over the elements of S_n can be reduced, as for a finite group (5.1), to a sum over the classes of S_n where each class is weighted by its order.

For each class of S_n there still remains an integral over the eigenvalues $\boldsymbol{\epsilon}_\lambda$ of $SU(2)$ and the angles $\boldsymbol{\theta}_\lambda$. These integrals require a volume element for the region of $\mathcal{H}$ that is to be integrated over. The angles label elements of the torus T, each set labels a single element which are all weighted equally. So to integrate with respect to the angle θ_{λ_j} we use the infinitesimal angle $d\theta_{\lambda_j}$. The condition on the angles $e^{i\sum\theta_{\lambda_j}} = \mathrm{sgn}(\rho)$ reduces the number of angles that we need to integrate over by one. So if their are r cycles in the class of S_n there will be $r-1$ integrals.

The parameters $\boldsymbol{\epsilon}_\lambda$ are eigenvalues of $SU(2)$ and for each choice of ϵ_j there are many matrices in $SU(2)$ with the required eigenvalue. The eigenvalues have modulus one so we can write $\epsilon_j = e^{i\phi_j}$ where ϕ_j runs from zero to 2π. The infinitesimal volume element for $SU(2)$ is $\Delta_j\overline{\Delta}_j\, d\phi_j$ where

$$\Delta_j = \begin{vmatrix} \epsilon_j & \overline{\epsilon}_j \\ 1 & 1 \end{vmatrix} \tag{5.69}$$

see section 2.7. Alternatively we can write

$$\Delta_j \overline{\Delta}_j = (\epsilon_j - \overline{\epsilon}_j)(\overline{\epsilon}_j - \epsilon_j)$$

We see the two eigenvalues of $SU(2)$ appear symmetrically in the volume element.

We can now write the integral of a product of two characters over the group $\mathcal{H}$. To correspond to the character decompositions in (5.1) and (5.2) we will take the character $X_{SU(2n)}$ to be a reducible character of a representation of $SU(2n)$ restricted to elements of the subgroup $\mathcal{H}$ and $\chi_{\mathcal{H}}$ to be an irreducible character of $\mathcal{H}$, then

$$N_{\chi}^{X} = \frac{1}{\Omega_{S_n}} \sum_{\lambda} \Omega_{\lambda S_n} A_{\lambda} \tag{5.70}$$

where

$$A_{\lambda} = \frac{1}{\Omega_{\lambda\mathcal{H}}} \int_0^{2\pi} \cdots \int_0^{2\pi} \Delta\overline{\Delta} d\theta_{\lambda_1} \dots d\theta_{\lambda_{r-1}} d\phi_1 \dots d\phi_r$$
$$\overline{X}_{SU(2n)}(\lambda, \boldsymbol{\theta}_{\lambda}, \boldsymbol{\epsilon}_{\lambda}) \chi_{\mathcal{H}}(\lambda, \boldsymbol{\theta}_{\lambda}, \boldsymbol{\epsilon}_{\lambda}) \tag{5.71}$$

and

$$\Delta\overline{\Delta} = \Delta_1\overline{\Delta}_1 . \Delta_2\overline{\Delta}_2 \dots \Delta_r\overline{\Delta}_r$$

This is the form of the character orthogonality relations we will use later to decompose the irreducible representations of $SU(2n)$. The volume $\Omega_{\lambda\mathcal{H}}$ is found by integrating the infinitesimal volume elements as with the unitary group in theorem 2.7.1.

$$\Omega_{\lambda\mathcal{H}} = \int_0^{2\pi} \cdots \int_0^{2\pi} \Delta\overline{\Delta} d\theta_{\lambda_1} \dots d\theta_{\lambda_{r-1}} d\phi_1 \dots d\phi_r \tag{5.72}$$

Each of the $r-1$ angle integrals with respect to θ contribute a factor of 2π. An integral with respect to ϕ produces a factor of $\Omega_{SU(2)}$ which when evaluated turns out to be 2.

$$\Omega_{\lambda\mathcal{H}} = (2\pi)^{r-1} 2^r \tag{5.73}$$

Using these results we will be able to determine the decomposition of a representation of $SU(2n)$ given the characters of the representations of $SU(2n)$ and $\mathcal{H}$.

5.4.5 Representations of $\mathcal{H}$

As $\mathcal{H}$ is a semidirect product we can apply the general theory developed in section 5.2.2 to find the irreducible representations of $\mathcal{H}$.

$$\mathcal{H} = \Sigma_n \ltimes [SU(2)]^n \tag{5.74}$$

Σ_n is also a semidirect product, $\Sigma_n = S_n \ltimes T^{n-1}$ and it is the group S_n that defines the automorphisms of both T^{n-1} and $[SU(2)]^n$. From this we can refine the definition of $\mathcal{H}$

$$\mathcal{H} = S_n \ltimes (T^{n-1} \times [SU(2)]^n) \tag{5.75}$$

Irreducible representations of $\mathcal{H}$ are labelled by irreducible representations of $T^{n-1} \times [SU(2)]^n$ and projective representations of the stabiliser of the representation of $T^{n-1} \times [SU(2)]^n$ in S_n.

An irreducible representation R of $[SU(2)]^n$ was defined in (5.61)

$$R(\mathbf{u}) = R^1(u_1) \otimes R^2(u_2) \otimes \cdots \otimes R^n(u_n) \tag{5.76}$$

$R^j(u)$ is an irreducible representation of $SU(2)$. The irreducible representations Q of T^{n-1} are labelled by the vector of integers $\mathbf{m}$. From 5.55

$$Q^{\mathbf{m}}(\boldsymbol{\theta}) = e^{i \sum m_j \theta_j} \tag{5.77}$$

Together $Q^{\mathbf{m}}(\boldsymbol{\theta})$ and $R(\mathbf{u})$ define an irreducible representation of $[SU(2)]^n \times T^{n-1}$. The group of automorphisms S_n defines maps between irreducible representations. For an element ρ of S_n

$$R_\rho(\mathbf{u}) = R(\rho^{-1}(\mathbf{u})) \tag{5.78}$$

Similarly

$$Q^{\rho(\mathbf{m})}(\boldsymbol{\theta}) = Q^{\mathbf{m}}(\rho^{-1}(\boldsymbol{\theta})) \tag{5.79}$$

These two relations define maps between the irreducible representations of $[SU(2)]^n \times T^{n-1}$.

The stabiliser under S_n of the representation of $[SU(2)]^n \times T^{n-1}$ labelled by R and $\mathbf{m}$ consists of the elements of S_n which map the representation $R, \mathbf{m}$ into itself.

$$S_{n,R,\mathbf{m}} = \{\rho \in S_n : R_\rho = R \text{ and } \rho(\mathbf{m}) = \mathbf{m}\} \tag{5.80}$$

If all the R^j in R are different then clearly the stabiliser contains only the identity element. This applies equally if all the m_j in $\mathbf{m}$ are different.

We can divide the n symbols into sets $\{i, j, \ldots, k\}$ where $R^i = R^j = \cdots = R^k$ and $m_i = m_j = \cdots = m_k$. Then if ρ is an element of S_n which only permutes symbols in the same set we know that $R_\rho = R$ and $\rho(\mathbf{m}) = \mathbf{m}$ so ρ is in the stabiliser of $R, \mathbf{m}$. The permutation group on a set of symbols $\{i, j, \ldots, k\}$ is a subgroup of S_n. We define $S_{n,R,\mathbf{m}}$ to be the direct product of the permutation groups on all such sets of symbols $\{i, j, \ldots, k\}$. Then $S_{n,R,\mathbf{m}}$ is the stabiliser of the representation $R, \mathbf{m}$ of $[SU(2)]^n \times T^{n-1}$. As $S_{n,R,\mathbf{m}}$ is formed from subgroups of S_n we can see that it is also a subgroup of S_n.

To clarify this let us take an example for $n = 6$. If we choose a representation of $[SU(2)]^n \times T^{n-1}$ where

$$R^1 = R^2 = R^3 \quad R^4 = R^5 = R^6$$
$$m_1 = m_2 \qquad m_4 = m_5 = m_6$$

Then the stabiliser is the product of the symmetric groups on the symbols $\{1, 2\}$ and $\{4, 5, 6\}$. Any ρ in this subgroup will map the representation $R, \mathbf{m}$ back into itself.

If there are q sets of symbols $\{i, j, \ldots, k\}$ then an irreducible representation of $S_{n,R,\mathbf{m}}$ is labelled by q partitions $\lambda^1 \ldots \lambda^q$ where each defines an irreducible representation of the permutation group on one set $\{i, j, \ldots, k\}$. We can use this to define an irreducible representation of $\mathcal{H}$. The representation is labelled by a representation R of $[SU(2)]^n$, a representation $\mathbf{m}$ of T^{n-1} and a representation $\lambda^1, \ldots, \lambda^q$ of $S_{n,R,\mathbf{m}}$. These representations will be sufficient for our problem without considering projective representations of the stabiliser.

We can use the general results for the semidirect product in section 5.2.2 to define how such a representation of $\mathcal{H}$ acts on basis vectors of the carrier space of the representation. Let $S_{n,R,\mathbf{m}}$ be the stabiliser of $R(\mathbf{u}) \otimes Q^{\mathbf{m}}(\boldsymbol{\theta})$. $\Gamma^{\lambda^1 \ldots \lambda^r}(S_{n,R,\mathbf{m}})$ is an irreducible representation of the stabiliser. A vector in the representation of $\mathcal{H}$ can be written

$$|h\rangle = |b\rangle \otimes c \otimes |\mathbf{v}\rangle \otimes |\nu\rangle \tag{5.81}$$

where c is a complex number of modulus one, the space acted on by the one dimensional representation $Q^{\mathbf{m}}$. $|\mathbf{v}\rangle$ is a basis vector of the carrier space of R so it can be written as a tensor product of n basis vectors of the representations R_j,

$$|\mathbf{v}\rangle = |v_1\rangle \otimes |v_2\rangle \otimes \cdots \otimes |v_n\rangle$$

The basis vector $|\nu\rangle$ is in the carrier space of the representation $\Gamma^{\lambda^1 \ldots \lambda^q}(S_{n,R,\mathbf{m}})$ and $|b\rangle$ is defined by the coset representatives σ_b of $S_n/S_{n,R,\mathbf{m}}$.

From equations (5.33) and (5.34) an irreducible representation of $\mathcal{H}$ is defined by

$$(I_{S_n}, \boldsymbol{\theta}, I_{[SU(2)]^n})\,|h\rangle \;=\; |b\rangle \otimes e^{i\sum m_j \theta_{\sigma_b(j)}}\, c \otimes |\mathbf{v}\rangle \otimes |\nu\rangle \tag{5.82}$$

$$(I_{S_n}, I_{T^{n-1}}, \mathbf{u})\,|h\rangle \;=\; |b\rangle \otimes c \otimes R(\sigma_b(\mathbf{u}))|\mathbf{v}\rangle \otimes |\nu\rangle \tag{5.83}$$

$$(\rho, I_{T^{n-1}}, I_{[SU(2)]^n})\,|h\rangle \;=\; |d\rangle \otimes c \otimes |\alpha^{-1}(\mathbf{v})\rangle \otimes \Gamma^{\lambda^1 \ldots \lambda^q}(\alpha)|\nu\rangle \tag{5.84}$$

where $\rho\sigma_b = \sigma_d \alpha$ with $\alpha \in S_{n,R,\mathbf{m}}$.

5.4.6 Physical representations of $\mathcal{H}$

Using the semidirect product we have classified the irreducible representations of $\mathcal{H}$. The vectors of $SU(2n)$ eligible for use in the construction can belong only to particular representations of the subgroup $\mathcal{H}$. These are the representations that we will now determine.

From **5.1.1** vectors used in the construction must have equal spins s with respect to the n spin subgroup. Restricting an irreducible representation of $\mathcal{H}$, labelled by R,

$\mathbf{m}$ and $\lambda^1 \dots \lambda^q$, to the $[SU(2)]^n$ subgroup produces the representation $R([SU(2)]^n)$, see equation (5.83). The n spins are identical if the representations $R^j(SU(2))$ used to construct R are the same.

$$R^1 = R^2 = \cdots = R^n \tag{5.85}$$

This is the first condition on the physical irreducible representations of $\mathcal{H}$ which can be used to generate a position dependent spin basis.

Spin vectors used in the construction are also zero weight vectors of the exchange angular momentum, condition **5.1.2**. This ensures that the Weyl group acts as the permutation group on the n spins and that a representation of Σ_n descends to the Weyl group. For this to be the case we saw that a representation $\Gamma(\Sigma_n)$ of the group of automorphisms of the exchange angular momentum can not depend on the phases $\boldsymbol{\theta}$. From equation (5.59)

$$\Gamma((\rho, \boldsymbol{\theta}))\,|\mathbf{M}\rangle = \Gamma((\rho, \boldsymbol{\theta}))\,\Gamma((I, \boldsymbol{\psi}))\,|\mathbf{M}\rangle \tag{5.86}$$

If we restrict a physical representation of $\mathcal{H}$ to the subgroup Σ_n then the representation of Σ_n we obtain should obey the condition (5.86). From (5.82) we see that for this to be the case the representation of $\mathcal{H}$ must be constructed from the trivial representation of T^{n-1},

$$m_1 = m_2 = \cdots = m_n = 0 \tag{5.87}$$

From the two conditions (5.85) and (5.87) on the representations R and $\mathbf{m}$ we can determine the stabiliser, $S_{n,R,\mathbf{m}}$, for these physical representations of $\mathcal{H}$.

$$S_{n,R,\mathbf{m}} = \{\rho \in S_n : R_\rho = R \text{ and } \rho(\mathbf{m}) = \mathbf{m}\} = S_n \tag{5.88}$$

All elements of S_n map a representation R with n identical spins back into itself and map the trivial representation of T^{n-1} to the trivial representation. As irreducible representations of $\mathcal{H}$ are labelled by representations of $S_{n,R,\mathbf{m}}$ the physical representations are labelled by an irreducible representation of S_n. A representation of S_n is labelled by a single partition λ. Selecting the representation λ of S_n can be seen as a choice of representation of the Weyl group which permutes the spins in the construction. The physical representations of $\mathcal{H}$ are labelled by a choice of spin s, which defines the $R_j(SU(2))$ in $R([SU(2)]^n)$ and a representation λ of S_n.

Rewriting equations (5.82) to (5.84) for physical representations of $\mathcal{H}$ we obtain

$$(I_{S_n}, \boldsymbol{\theta}, I_{[SU(2)]^n})\,|\mathbf{v}\rangle \otimes |\nu\rangle \;=\; |\mathbf{v}\rangle \otimes |\nu\rangle \tag{5.89}$$

$$(I_{S_n}, I_{T^{n-1}}, \mathbf{u})\,|\mathbf{v}\rangle \otimes |\nu\rangle \;=\; R(\mathbf{u})|\mathbf{v}\rangle \otimes |\nu\rangle \tag{5.90}$$

$$(\rho, I_{T^{n-1}}, I_{[SU(2)]^n})\,|\mathbf{v}\rangle \otimes |\nu\rangle \;=\; |\rho^{-1}(\mathbf{v})\rangle \otimes \Gamma^\lambda(\rho)|\nu\rangle \tag{5.91}$$

In defining the vectors in the carrier space of the representation the constant c has been removed. As we are considering only the trivial representation of T^{n-1} its inclusion would serve no purpose. For the physical representations of $\mathcal{H}$ the quotient group $S_n/S_{n,R,\mathbf{m}}$ is S_n/S_n. This is a group of one element and so there is only a single vector $|b\rangle$ which is also omitted. As the stabiliser $S_{n,R,\mathbf{m}}$ is now the whole of S_n we do not need to define a separate element α of $S_{n,R,\mathbf{m}}$. Equation (5.91) defines a representation P of the symmetric group where,

$$P(\rho)|v_1\rangle \otimes |v_2\rangle \otimes \cdots \otimes |v_n\rangle = |v_{\rho^{-1}(1)}\rangle \otimes |v_{\rho^{-1}(2)}\rangle \otimes \cdots \otimes |v_{\rho^{-1}(n)}\rangle \tag{5.92}$$

P is a representation which permutes the tensor product.

Equations (5.89), (5.90) and (5.91) define the physical irreducible representations of $\mathcal{H}$. In order to investigate further the properties of these representations it will be useful to construct them explicitly. A physical representation $B^{\lambda s}(\mathcal{H})$ is

$$B^{s\lambda}(\rho, \boldsymbol{\theta}, \mathbf{u}) = [(R^s(u_1) \otimes \cdots \otimes R^s(u_n))\, P(\rho)] \otimes \Gamma^\lambda(\rho) \tag{5.93}$$

It is the number and type of these representations $B^{s\lambda}$ in a representation of $SU(2n)$ restricted to $\mathcal{H}$ that we want to determine. A spin-statistics connection is a relationship between the spin and the representation λ of S_n for particular choices of representations of $SU(2n)$. The representation λ of S_n determines how spin vectors transform under permutations of the spins. If λ is the trivial representation spin vectors are symmetric when permuted, a property which then transfers to the position dependent spin basis. The alternating representation of the symmetric group provides antisymmetric spin vectors and other representations of S_n correspond to spin vectors which produce parastatistics in the position-dependent basis.

5.4.7 Characters of $B^{s\lambda}(\mathcal{H})$

We intend to use the character orthogonality relations to decompose a representation $\Gamma(SU(2n))$ restricted to $\mathcal{H}$. To apply the character orthogonality relations we must know the characters of the irreducible representations of $\mathcal{H}$ whose presence we want to determine in Γ, these are the characters of the physical representations $B^{s\lambda}(\mathcal{H})$.

The representation $B^{s\lambda}(\mathcal{H})$ defined in (5.93) is the tensor product of a representation $\Gamma^{\lambda}(S_n)$ and the representation $[(R^s(u_1) \otimes \cdots \otimes R^s(u_n))\, P(\rho)]$ of $\mathcal{H}$. The character of a representation is the trace of the representation and the trace of a tensor product of two matrices is the product of the traces of the two matrices. Therefore to find the character of $B^{s\lambda}(\mathcal{H})$ it is sufficient to know the characters of the representations $\Gamma^{\lambda}(S_n)$ and $[(R^s(u_1) \otimes \cdots \otimes R^s(u_n))\, P(\rho)]$ and take their product. Γ^{λ} is an irreducible representation of S_n for which the characters are well known, see section 2.8. It remains to determine the trace of matrices $[(R^s(u_1) \otimes \cdots \otimes R^s(u_n))\, P(\rho)]$.

As a preliminary we will state a lemma for the trace of a tensor product multiplied by $P(\rho)$.

Lemma 5.4.1. *For any n $m \times m$ matrices $U^1, \ldots, U^n$*

$$Tr[(U^1 \otimes \ldots \otimes U^n)\, P(\rho)] = \prod_{cycles\ of\ \rho} Tr(U^i U^j \ldots U^l)$$

where $P(\sigma)$ permutes the tensor product of n vectors and $(ijk \ldots l)$ is a cycle in ρ.

Proof: Let U^j be a matrix with elements $U^j_{p_j q_j}$. $P(\rho)$ acting to the left permutes columns in $U^1 \otimes \ldots \otimes U^n$ so

$$[(U^1 \otimes \ldots \otimes U^n)\, P(\rho)]_{p_1 \ldots p_n\, q_1 \ldots q_n} = U^1_{p_1 q_{\rho^{-1}(1)}} \ldots U^n_{p_n q_{\rho^{-1}(n)}} \tag{5.94}$$

Setting $p_i = q_i$ and summing

$$Tr[(U^1 \otimes \ldots \otimes U^n)\, P(\rho)] = \sum_{q_1 \ldots q_n} U^1_{q_1 q_{\rho^{-1}(1)}} \ldots U^n_{q_n q_{\rho^{-1}(n)}} \tag{5.95}$$

Now let $(ijk\dots l)$ be a cycle in ρ so we know that $\rho^{-1}(i) = j$. Taking the terms in (5.95) with these labels

$$\sum_{q_i q_j \dots q_l} U^i_{q_i q_j} U^j_{q_j q_k} \dots U^l_{q_l q_i} = Tr\left(U^i U^j \dots U^l\right) \qquad \square$$

Using lemma 5.4.1 the characters of $[(R^s(u_1)\otimes\cdots\otimes R^s(u_n))\,P(\rho)]$ are the product on the cycles of ρ of the characters of $R^s(u_{ijk\dots l})$, where

$$u_{ijk\dots l} = u_i u_j u_k \dots u_l \tag{5.96}$$

The representation $R^s(SU(2))$ can also be labelled by the integer $2s$, this is the number of boxes in the Young tableau of one row which distinguish irreducible representations of $SU(2)$. The character of such a representation of $SU(2)$ is given by the Weyl character formula, section 2.7.

$$\chi(R_s(u)) = \frac{\begin{vmatrix} \epsilon^{2s} & 1 \\ \overline{\epsilon}^{2s} & 1 \end{vmatrix}}{\begin{vmatrix} \epsilon & 1 \\ \overline{\epsilon} & 1 \end{vmatrix}} = \frac{\epsilon^{2s} - \overline{\epsilon}^{2s}}{\epsilon - \overline{\epsilon}} \tag{5.97}$$

ϵ and $\overline{\epsilon}$ are the eigenvalues of u. They label the classes of $SU(2)$.

If we combine these results we can write the character $\chi^s_{\mathcal{H}}$ of the representations $[(R^s(u_1)\otimes\cdots\otimes R^s(u_n))\,P(\rho)]$ of $\mathcal{H}$,

$$\chi^s_{\mathcal{H}}(\rho, \boldsymbol{\theta}, \mathbf{u}) = \prod_{\text{cycles of } \rho} \frac{\begin{vmatrix} \epsilon^{2s} & 1 \\ \overline{\epsilon}^{2s} & 1 \end{vmatrix}}{\begin{vmatrix} \epsilon & 1 \\ \overline{\epsilon} & 1 \end{vmatrix}} \tag{5.98}$$

where ϵ and $\overline{\epsilon}$ are the eigenvalues of $u_{ij\dots l}$. It follows that there is one eigenvalue ϵ for each cycle in ρ. (As the character is a class function this agrees with the definition of the classes of $\mathcal{H}$. A class of $\mathcal{H}$ is labelled by a class of S_n with one angle θ and one eigenvalue ϵ for each cycle in the class of S_n.) The characters of the physical representations $B^{s\lambda}(\mathcal{H})$ are then given by the product of characters

$$\chi^{B^{s\lambda}}_{\mathcal{H}}(\rho, \boldsymbol{\theta}, \mathbf{u}) = \chi^s_{\mathcal{H}}(\rho, \boldsymbol{\theta}, \mathbf{u})\, \chi^{\lambda}_{S_n}(\rho) \tag{5.99}$$

The physical representations are independent of the angles $\boldsymbol{\theta}$ so the angles do not appear in the characters of $B^{s\lambda}(\mathcal{H})$. The characters are still functions of the eigenvalues $\boldsymbol{\epsilon}_\kappa$ and the class κ of ρ in S_n.

5.5 Representations of $SU(2n)$

The irreducible representations of $SU(2n)$ are labelled by a vector of $2n-1$ integers $\mathbf{f} = (f_1, \ldots, f_{2n-1})$. The integers are the lengths of the rows of the Young tableau associated with the representation. Classes of $SU(2n)$ are labelled by $2n$ complex numbers of modulus one, $\boldsymbol{\epsilon} = (\epsilon_1, \ldots, \epsilon_{2n})$ with the condition that their product is unity. We will often think of this condition as a definition of ϵ_{2n}

$$\overline{\epsilon}_{2n} = \epsilon_1.\epsilon_2 \ldots \epsilon_{2n-1} \tag{5.100}$$

The complex numbers $\boldsymbol{\epsilon}$ are the eigenvalues of the elements of $SU(2n)$ and the classes of an element u of $SU(2n)$ is determined by its diagonal matrix of eigenvalues

$$\begin{pmatrix} \epsilon_1 & & & \\ & \epsilon_2 & & \\ & & \ddots & \\ & & & \epsilon_{2n} \end{pmatrix} = \{\epsilon_1, \epsilon_2, \ldots, \epsilon_{2n}\} \tag{5.101}$$

The order of the terms ϵ_i in $\boldsymbol{\epsilon}$ is therefore arbitrary.

5.5.1 Characters of $SU(2n)$

The characters of $SU(2n)$ are a function of the class $\boldsymbol{\epsilon}$. Using the Weyl formula we can write the irreducible characters as a ratio of determinants

$$X^{\mathbf{f}}_{SU(2n)}(\boldsymbol{\epsilon}) = \frac{\begin{vmatrix} \epsilon_1^{f_1+(2n-1)} & \epsilon_2^{f_1+(2n-1)} & \cdots & \epsilon_{2n}^{f_1+(2n-1)} \\ \epsilon_1^{f_2+(2n-2)} & \epsilon_2^{f_2+(2n-2)} & \cdots & \epsilon_{2n}^{f_2+(2n-2)} \\ \vdots & \vdots & & \vdots \\ \epsilon_1^{f_{2n-1}} & \epsilon_2^{f_{2n-1}} & \cdots & \epsilon_{2n}^{f_{2n-1}} \\ 1 & 1 & \cdots & 1 \end{vmatrix}}{\begin{vmatrix} \epsilon_1^{(2n-1)} & \epsilon_2^{(2n-1)} & \cdots & \epsilon_n^{(2n-1)} \\ \vdots & \vdots & & \vdots \\ \epsilon_1 & \epsilon_2 & \cdots & \epsilon_n \\ 1 & 1 & \cdots & 1 \end{vmatrix}} \tag{5.102}$$

As these ratios are awkward to write we will abbreviate the Vandemonde determinants by writing only the first row of the matrix. So for example

$$|\epsilon_1^{f_1+(2n-1)} \ldots \epsilon_n^{f_1+(2n-1)}| \quad \equiv \begin{vmatrix} \epsilon_1^{f_1+(2n-1)} & \cdots & \epsilon_n^{f_1+(2n-1)} \\ \epsilon_1^{f_2+(2n-2)} & \cdots & \epsilon_n^{f_2+(2n-2)} \\ \vdots & & \vdots \\ \epsilon_1^{f_n} & \cdots & \epsilon_n^{f_n} \end{vmatrix} \tag{5.103}$$

Using this notation the character of an irreducible representation of $SU(2n)$ is

$$X^{\mathbf{f}}_{SU(2n)}(\boldsymbol{\epsilon}) = \frac{|\, \epsilon_1^{f_1+(2n-1)} \ldots \epsilon_{2n}^{f_1+(2n-1)} \,|}{|\, \epsilon_1^{(2n-1)} \ldots \epsilon_n^{(2n-1)} \,|} \tag{5.104}$$

5.5.2 Restricting a representation of $SU(2n)$ to $\mathcal{H}$

To write down the character of $SU(2n)$ for an element $h \in \mathcal{H}$ we must find the eigenvalues of h. From equation (5.62) an element in $\mathcal{H}$ can be factorised

$$h = \begin{pmatrix} e^{i\theta_1}u_1 & & \\ & \ddots & \\ & & e^{i\theta_n}u_n \end{pmatrix} (D(\rho) \otimes I) \tag{5.105}$$

When ρ is the identity the eigenvalues are $e^{i\theta_1}\epsilon_1, e^{i\theta_1}\overline{\epsilon}_1, \ldots, e^{i\theta_n}\epsilon_n, e^{i\theta_n}\overline{\epsilon}_n$, where ϵ_j is an eigenvalue of u_j.

We take a vector $\mathbf{v}$ to be an eigenvector of h with eigenvalue τ.

$$\begin{pmatrix} e^{i\theta_1}u_1 & & \\ & \ddots & \\ & & e^{i\theta_n}u_n \end{pmatrix} (D(\rho) \otimes I)\mathbf{v} = \tau\mathbf{v} \tag{5.106}$$

We will treat $\mathbf{v}$ as the direct sum of n two-dimensional vectors

$$\mathbf{v} = \mathbf{v}_1 \oplus \mathbf{v}_2 \oplus \cdots \oplus \mathbf{v}_n \tag{5.107}$$

If ρ is not the identity let $(jk\ldots l)$ be an m cycle in ρ. For this cycle we obtain, from the eigenvector equation (5.106), m equations

$$\begin{aligned} e^{i\theta_j}u_j\,\mathbf{v}_j &= \tau\,\mathbf{v}_k \\ &\vdots \\ e^{i\theta_l}u_l\,\mathbf{v}_l &= \tau\,\mathbf{v}_j \end{aligned} \tag{5.108}$$

These can be combined in order to find an eigenvector equation for $\mathbf{v}_k$

$$e^{i(\theta_j+\theta_k+\cdots+\theta_l)}u_j u_l \ldots u_k\,\mathbf{v}_k = \lambda^m\,\mathbf{v}_k \tag{5.109}$$

Similar equations are obtained for the other $\mathbf{v}_j$ in the cycle. In each case the elements u in the product have undergone a cyclic permutation. If $\mathbf{v}_k$ is an eigenvector of $u_j u_l \ldots u_k$, which we will call $u_{jl\ldots k}$, with eigenvalue ϵ then $\mathbf{v}_k$ will also satisfy (5.109) with

$$\tau = e^{2\pi ip/m}\, e^{i(\theta_j+\cdots+\theta_l)/m}\, \epsilon^{\frac{1}{m}} \tag{5.110}$$

where p is an integer. τ is an m'th root of unity multiplied by a phase determined by the sum of the phases in the cycle and an m'th root of the eigenvalue ϵ of $u_{jk\ldots l}$. A cyclic permutation of the u's will have the same eigenvalue ϵ so each version of equation (5.109) will yield the same values of τ which is as we require. By setting vectors $\mathbf{v}_i$ not in the cycle equal to the zero vector we find eigenvectors $\mathbf{v}$ whose eigenvalues are given by the relation (5.110) for each of the cycles in ρ. As $\overline{\epsilon}$ is also an eigenvalue of $u_{jk\ldots l}$ and there are m roots of unity, each m cycle contributes $2m$ eigenvalues. Summing over the cycles in ρ we obtain the $2n$ eigenvalues of h. The characters of $SU(2n)$ restricted to the subgroup $\mathcal{H}$ are then found by plugging the eigenvalues (5.110) of h into the Weyl character formula (5.102). The eigenvalues ϵ and the sum of the phases in a cycle, $\theta_{jk\ldots l} = \theta_j + \theta_k + \cdots + \theta_l$ were the parameters used in section 5.4.3 to distinguish classes of $\mathcal{H}$.

5.6 The decomposition of $SU(4)$

The two spin example is the simplest character decomposition and the calculations in this case can be done most explicitly. The results can also be compared to those obtained using Young tableau.

In this section we will only be considering the case where $n = 2$. The permutation group S_2 contains two elements, I and (12). $\mathcal{H}$ contains two disjoint components depending on the element of S_2 used to construct h in $\mathcal{H}$ and the classes of $\mathcal{H}$ are labelled by different parameters in these regions. For the class I of S_2 two parameters are eigenvalues of $SU(2)$, ϵ_1 and ϵ_2, there is also an angle θ_1. The second angle θ_2 used to construct elements of $\mathcal{H}$ is determined by the condition, $e^{i(\theta_1+\theta_2)} = \mathrm{sgn}(I) = 1$. When ρ is (12) there is only one parameter, the eigenvalue ϵ_{12} of u_1u_2. The angle $\theta_{12} = \theta_1 + \theta_2$ is determined by the condition $e^{i\omega} = \mathrm{sgn}(\rho) = -1$. To sum the product of characters of $SU(4)$ and $\mathcal{H}$ over the classes of $\mathcal{H}$ we must integrate over the two continuous regions and sum the results.

An irreducible representation of $SU(4)$ is labelled by $\mathbf{f} = (f_1, f_2, f_3)$ and the corresponding character is $X^{\mathbf{f}}_{SU(4)}$. From equations (5.70) and (5.71) for the character decomposition of a reducible representation of $\mathcal{H}$ the number of physical representations $B_{s\lambda}(\mathcal{H})$ in a representation $\mathbf{f}$ of $SU(4)$ is given by

$$N^{\mathbf{f}}_{s\lambda} = \frac{1}{2}A_I + \frac{\chi^{\lambda}_{S_2}((12))}{2}A_{12} \tag{5.111}$$

$$\begin{aligned} A_I &= \frac{1}{\Omega_I}\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} \Delta\overline{\Delta}\, d\phi_1 d\phi_2 d\theta_1 \\ &\quad \overline{X}^{\mathbf{f}}_{SU(4)}(I, \boldsymbol{\epsilon}_I, \boldsymbol{\theta}_I)\chi^{s}_{\mathcal{H}}(I, \boldsymbol{\epsilon}_I) \end{aligned} \tag{5.112}$$

$$A_{12} = \frac{1}{\Omega_{12}}\int_0^{2\pi} \overline{X}^{\mathbf{f}}_{SU(4)}((12), \boldsymbol{\epsilon}_{12}, \boldsymbol{\theta}_{12})\chi^{s}_{\mathcal{H}}((12), \boldsymbol{\epsilon}_{12})\, \Delta\overline{\Delta}\, d\phi_{12} \tag{5.113}$$

The factors of 1/2 are from the order of S_2. $\chi^{\lambda}_{S_2}((12))$ is ± 1 depending on the whether the representation $B_{s\lambda}(\mathcal{H})$ is constructed from the trivial or alternating representation of S_2. For example to find the number of representations where the vectors are antisymmetric under the permutation of the two spins A_{12} is subtracted from A_I and the result is divided by two. In equations (5.112) and (5.113) $X^{\mathbf{f}}_{SU(4)}(\kappa, \boldsymbol{\epsilon}_\kappa, \boldsymbol{\theta}_\kappa)$ is the character of the representation $\mathbf{f}$ of $SU(4)$ restricted to the classes of $\mathcal{H}$ labelled by κ.

5.6.1 The character of $SU(4)$ for elements of $\mathcal{H}$

The eigenvalues of an element of $\mathcal{H}$ are given by equation (5.110). The two classes of S_2 are one two-cycle, (12), or two one-cycles, I. For I the four eigenvalues are

$$e^{i\theta_1}\epsilon_1 \quad e^{i\theta_1}\overline{\epsilon}_1 \quad e^{-i\theta_1}\epsilon_2 \quad e^{-i\theta_1}\overline{\epsilon}_2$$

ϵ_1 and ϵ_2 are eigenvalues of u_1 and u_2 respectively. These eigenvalues can be substituted into the Weyl formula (5.102) for the character of the representation $\mathbf{f}$ of $SU(4)$

$$X^{\mathbf{f}}_{SU(4)}(I, \boldsymbol{\theta}_I, \boldsymbol{\epsilon}_I) = \frac{\begin{vmatrix} (e^{i\theta_1}\epsilon_1)^{f_1+3} & (e^{i\theta_1}\overline{\epsilon}_1)^{f_1+3} & (e^{-i\theta_1}\epsilon_2)^{f_1+3} & (e^{-i\theta_1}\overline{\epsilon}_2)^{f_1+3} \\ (e^{i\theta_1}\epsilon_1)^{f_2+2} & (e^{i\theta_1}\overline{\epsilon}_1)^{f_2+2} & (e^{-i\theta_1}\epsilon_2)^{f_2+2} & (e^{-i\theta_1}\overline{\epsilon}_2)^{f_2+2} \\ (e^{i\theta_1}\epsilon_1)^{f_3+1} & (e^{i\theta_1}\overline{\epsilon}_1)^{f_3+1} & (e^{-i\theta_1}\epsilon_2)^{f_3+1} & (e^{-i\theta_1}\overline{\epsilon}_2)^{f_3+1} \\ 1 & 1 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} (e^{i\theta_1}\epsilon_1)^{3} & (e^{i\theta_1}\overline{\epsilon}_1)^{3} & (e^{-i\theta_1}\epsilon_2)^{3} & (e^{-i\theta_1}\overline{\epsilon}_2)^{3} \\ (e^{i\theta_1}\epsilon_1)^{2} & (e^{i\theta_1}\overline{\epsilon}_1)^{2} & (e^{-i\theta_1}\epsilon_2)^{2} & (e^{-i\theta_1}\overline{\epsilon}_2)^{2} \\ (e^{i\theta_1}\epsilon_1)^{1} & (e^{i\theta_1}\overline{\epsilon}_1)^{1} & (e^{-i\theta_1}\epsilon_2)^{1} & (e^{-i\theta_1}\overline{\epsilon}_2)^{1} \\ 1 & 1 & 1 & 1 \end{vmatrix}} \tag{5.114}$$

If we solve equation (5.110) for the eigenvalues of $\mathcal{H}$ using the two cycle (12) we obtain four different eigenvalues. Using the condition on the sum of the angles $e^{i\theta_{12}} = \mathrm{sgn}((12)) = -1$ to eliminate the angle θ_{12} the eigenvalues depend only on ϵ_{12},

$$i\epsilon_{12}^{1/2} \quad -i\epsilon_{12}^{1/2} \quad i\overline{\epsilon}_{12}^{1/2} \quad -i\overline{\epsilon}_{12}^{1/2}$$

Substituting these terms into the Weyl character formulae we obtain the character of $SU(4)$ for elements of $\mathcal{H}$ generated using (12).

$$X^{\mathbf{f}}_{SU(4)}((12),\boldsymbol{\theta}_{12},\boldsymbol{\epsilon}_{12}) = \frac{\begin{vmatrix} (i\epsilon_{12}^{1/2})^{f_1+3} & (i\overline{\epsilon}_{12}^{1/2})^{f_1+3} & (-i\epsilon_{12}^{1/2})^{f_1+3} & (-i\overline{\epsilon}_{12}^{1/2})^{f_1+3} \\ (i\epsilon_{12}^{1/2})^{f_2+2} & (i\overline{\epsilon}_{12}^{1/2})^{f_2+2} & (-i\epsilon_{12}^{1/2})^{f_2+2} & (-i\overline{\epsilon}_{12}^{1/2})^{f_2+2} \\ (i\epsilon_{12}^{1/2})^{f_3+1} & (i\overline{\epsilon}_{12}^{1/2})^{f_3+1} & (-i\epsilon_{12}^{1/2})^{f_3+1} & (-i\overline{\epsilon}_{12}^{1/2})^{f_3+1} \\ 1 & 1 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} (i\epsilon_{12}^{1/2})^{3} & (i\overline{\epsilon}_{12}^{1/2})^{3} & (-i\epsilon_{12}^{1/2})^{3} & (-i\overline{\epsilon}_{12}^{1/2})^{3} \\ (i\epsilon_{12}^{1/2})^{2} & (i\overline{\epsilon}_{12}^{1/2})^{2} & (-i\epsilon_{12}^{1/2})^{2} & (-i\overline{\epsilon}_{12}^{1/2})^{2} \\ (i\epsilon_{12}^{1/2}) & (i\overline{\epsilon}_{12}^{1/2}) & (-i\epsilon_{12}^{1/2}) & (-i\overline{\epsilon}_{12}^{1/2}) \\ 1 & 1 & 1 & 1 \end{vmatrix}} \tag{5.115}$$

5.6.2 Evaluation of A_I

The character $\chi^s_{\mathcal{H}}$ was defined in (5.98). Substituting the character formulae into the definition of A_I (5.112) we obtain the following integral

$$\begin{aligned} A_I \;=\; & \frac{1}{\Omega_I}\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} \Delta\overline{\Delta} \quad d\theta_1 d\phi_1 d\phi_2 \frac{\begin{vmatrix} \epsilon_1^{2s+1} & \overline{\epsilon}_1^{2s+1} \\ 1 & 1 \end{vmatrix}}{\begin{vmatrix} \epsilon_1 & \overline{\epsilon}_1 \\ 1 & 1 \end{vmatrix}} \frac{\begin{vmatrix} \epsilon_2^{2s+1} & \overline{\epsilon}_2^{2s+1} \\ 1 & 1 \end{vmatrix}}{\begin{vmatrix} \epsilon_2 & \overline{\epsilon}_2 \\ 1 & 1 \end{vmatrix}} \\ & \frac{\begin{vmatrix} (e^{-i\theta_1}\epsilon_1)^{f_1+3} & (e^{-i\theta_1}\overline{\epsilon}_1)^{f_1+3} & (e^{i\theta_1}\epsilon_2)^{f_1+3} & (e^{i\theta_1}\overline{\epsilon}_2)^{f_1+3} \\ (e^{-i\theta_1}\epsilon_1)^{f_2+2} & (e^{-i\theta_1}\overline{\epsilon}_1)^{f_2+2} & (e^{i\theta_1}\epsilon_2)^{f_2+2} & (e^{i\theta_1}\overline{\epsilon}_2)^{f_2+2} \\ (e^{-i\theta_1}\epsilon_1)^{f_3+1} & (e^{-i\theta_1}\overline{\epsilon}_1)^{f_3+1} & (e^{i\theta_1}\epsilon_2)^{f_3+1} & (e^{i\theta_1}\overline{\epsilon}_2)^{f_3+1} \\ 1 & 1 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} (e^{-i\theta_1}\epsilon_1)^{3} & (e^{-i\theta_1}\overline{\epsilon}_1)^{3} & (e^{i\theta_1}\epsilon_2)^{3} & (e^{i\theta_1}\overline{\epsilon}_2)^{3} \\ (e^{-i\theta_1}\epsilon_1)^{2} & (e^{-i\theta_1}\overline{\epsilon}_1)^{2} & (e^{i\theta_1}\epsilon_2)^{2} & (e^{i\theta_1}\overline{\epsilon}_2)^{2} \\ (e^{-i\theta_1}\epsilon_1)^{1} & (e^{-i\theta_1}\overline{\epsilon}_1)^{1} & (e^{i\theta_1}\epsilon_2)^{1} & (e^{i\theta_1}\overline{\epsilon}_2)^{1} \\ 1 & 1 & 1 & 1 \end{vmatrix}} \end{aligned} \tag{5.116}$$

where taking the complex conjugate of the character of $SU(4)$ in (5.114) changes the sign of θ_1, the effect on the eigenvalues ϵ is removed by permuting columns in the determinants. A change in the sign of the determinant in the numerator will be cancelled by the same sign change in the denominator.

To evaluate this integral we will employ the Littlewood-Richardson rule to express the $SU(4)$ character as a sum of $SU(2)$ characters. The same technique will be repeated in the general case. The coefficients found from the Littlewood-Richardson theorem give the multiplicity of irreducible of representations of $U(m) \times U(n)$ in the decomposition of an irreducible representation $U(m+n)$ this is discussed in section 2.9.3. As these coefficients are the multiplicities of representations they can also be applied to the decomposition of characters

$$\chi^{\mathbf{f}}_{U(m+n)} = \sum_{\boldsymbol{\alpha}\boldsymbol{\beta}} Y^{\mathbf{f}}_{\boldsymbol{\alpha}\boldsymbol{\beta}}\, \chi^{\boldsymbol{\alpha}}_{U(m)} \chi^{\boldsymbol{\beta}}_{U(n)} \tag{5.117}$$

$\mathbf{f}$ is a vector of $m+n$ integers labelling the irreducible representation of $U(m+n)$ and similarly $\boldsymbol{\alpha}$ is a vector of m integers and $\boldsymbol{\beta}$ a vector of n integers labelling the respective irreducible representations of $U(m)$ and $U(n)$. The coefficients $Y^{\mathbf{f}}_{\boldsymbol{\alpha}\boldsymbol{\beta}}$ can be evaluated using the rules for multiplying tableau.

The character of $U(m+n)$ is a function of the $m+n$ eigenvalues ϵ_1 to ϵ_{m+n}. We can take the first m eigenvalues, ϵ_1 to ϵ_m, to be the eigenvalues of the $U(m)$ subgroup and assign the remaining n eigenvalues to the $U(n)$ subgroup. By substituting the Weyl character formulae for the characters of the unitary group into equation (5.117) we can rewrite the Littlewood-Richardson decomposition,

$$\frac{|\epsilon_1^{f_1+(m+n-1)} \ldots \epsilon_{m+n}^{f_1+(m+n-1)}|}{|\epsilon_1^{(m+n-1)} \ldots \epsilon_{m+n}^{(m+n-1)}|} =$$

$$\sum_{\boldsymbol{\alpha}\boldsymbol{\beta}} Y^{\mathbf{f}}_{\boldsymbol{\alpha}\boldsymbol{\beta}} \quad \frac{|\epsilon_1^{\alpha_1+(m-1)} \ldots \epsilon_m^{\alpha_1+(m-1)}|}{|\epsilon_1^{(m-1)} \ldots \epsilon_m^{(m-1)}|} \frac{|\epsilon_{m+1}^{\beta_1+(n-1)} \ldots \epsilon_{m+n}^{\beta_1+(n-1)}|}{|\epsilon_{m+1}^{(n-1)} \ldots \epsilon_{m+n}^{(n-1)}|} \tag{5.118}$$

The determinants in the characters have been abbreviated as defined in (5.103). We notice that as we have dealt with the decomposition of the unitary group we have not needed to assume any relations between the eigenvalues. We can regard equation (5.118) as a factorisation of a the $m+n$ determinants into a sum of terms involving smaller determinants and we will refer to this as the Littlewood-Richardson factorisation. We should note that while using this formulae seems to imply a significant simplification in the evaluation of a large determinant, the rules for evaluating the coefficients $Y^{\mathbf{f}}_{\boldsymbol{\alpha\beta}}$ are complex. Effectively we are transferring some of the difficulty in evaluating the character into the evaluation of the coefficients.

We will now apply the Littlewood-Richardson factorisation (5.118) to the equation for A_I (5.116). We factorise the ratio of the determinants of 4×4 matrices into a sum of products of ratios of the determinants of 2×2 matrices.

$$\begin{aligned} A_I &= \frac{1}{\Omega_I} \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} \sum_{\boldsymbol{\alpha\beta}} Y^{\mathbf{f}}_{\boldsymbol{\alpha\beta}} \, \Delta\overline{\Delta} \, d\theta_1 d\phi_1 d\phi_2 \\ &\quad \frac{|(e^{-i\theta_1}\epsilon_1)^{\alpha_1+1} \; (e^{-i\theta_1}\overline{\epsilon}_1)^{\alpha_1+1}|}{|(e^{-i\theta_1}\epsilon_1) \quad (e^{-i\theta_1}\overline{\epsilon}_1)|} \frac{|(e^{i\theta_1}\epsilon_2)^{\beta_1+1} \; (e^{i\theta_1}\overline{\epsilon}_2)^{\beta_1+1}|}{|(e^{i\theta_1}\epsilon_2) \quad (e^{i\theta_1}\overline{\epsilon}_2)|} \\ &\quad \frac{|\epsilon_1^{2s+1} \; \overline{\epsilon}_1^{2s+1}|}{|\epsilon_1 \quad \overline{\epsilon}_1|} \frac{|\epsilon_2^{2s+1} \; \overline{\epsilon}_2^{2s+1}|}{|\epsilon_2 \quad \overline{\epsilon}_2|} \\ &= \frac{1}{\Omega_I} \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} \sum_{\boldsymbol{\alpha\beta}} Y^{\mathbf{f}}_{\boldsymbol{\alpha\beta}} \, \Delta\overline{\Delta} \, d\theta_1 d\phi_1 d\phi_2 \quad e^{-i(|\boldsymbol{\alpha}|\theta_1 - |\boldsymbol{\beta}|\theta_1)} \\ &\quad \frac{|\epsilon_1^{\alpha_1+1} \; \overline{\epsilon}_1^{\alpha_1+1}|}{|\epsilon_1 \quad \overline{\epsilon}_1|} \frac{|\epsilon_2^{\beta_1+1} \; \overline{\epsilon}_2^{\beta_1+1}|}{|\epsilon_2 \quad \overline{\epsilon}_2|} \frac{|\epsilon_2^{2s+1} \; \overline{\epsilon}_2^{2s+1}|}{|\epsilon_2 \quad \overline{\epsilon}_2|} \frac{|\epsilon_2^{2s+1} \; \overline{\epsilon}_2^{2s+1}|}{|\epsilon_2 \quad \overline{\epsilon}_2|} \end{aligned} \tag{5.119}$$

$|\boldsymbol{\alpha}|$ is the sum of the integers in $\boldsymbol{\alpha}$. The three integrals now separate and can be solved in turn.

The integral with respect to θ_1 is zero unless

$$|\boldsymbol{\alpha}| = |\boldsymbol{\beta}|$$

If A_I is not zero then the phase integral gives 2π which cancels with a factor of 2π in Ω_I. The integrals with respect to ϕ_1 and ϕ_2 are both integrals of the product of two irreducible characters of $SU(2)$ over the group $SU(2)$. Using the orthogonality of irreducible characters we know that these integrals will be zero unless the representations $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are both the spin s representation of $SU(2)$. A representation (α_1, α_2) of $SU(2)$ is equivalent to the representation $(\alpha_1 - \alpha_2, 0)$, this can be verified by looking at the Weyl character formula, therefore A_I is zero unless

$$\alpha_1 - \alpha_2 = 2s \qquad \beta_1 - \beta_2 = 2s$$

If we define a vector of integers $\mathbf{s} = (s_1, s_2)$ where $s_1 - s_2 = 2s$ then combining the conditions from the three integrals A_I is zero unless

$$\boldsymbol{\alpha} = \boldsymbol{\beta} = \mathbf{s} \tag{5.120}$$

If A_I is not zero the integrals with respect to ϕ_1 and ϕ_2 both produce factors of $\Omega_{SU(2)}$ which cancel the remaining terms in Ω_I. Only one term in the sum can have $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ which obey condition (5.120) and we have the result

$$A_I = Y^{\mathbf{f}}_{\mathbf{ss}} \tag{5.121}$$

$Y^{\mathbf{f}}_{\mathbf{ss}}$ is found by multiplying two identical tableau $\mathbf{s}$ and finding the number of tableau $\mathbf{f}$ in the result. This is equivalent to the result determined directly from the Young tableau in chapter 3 for the number of spin s subspaces with zero weight with respect to E_z.

5.6.3 Evaluation of A_{12}

We will apply a similar procedure to evaluate A_{12}.

$$\begin{aligned} A_{12} &= \frac{1}{\Omega_{12}} \int_0^{2\pi} \frac{|(i\epsilon_{12}^{1/2})^{f_1+3}\ (i\overline{\epsilon}_{12}^{1/2})^{f_1+3}\ (-i\epsilon_{12}^{1/2})^{f_1+3}\ (-i\overline{\epsilon}_{12}^{1/2})^{f_1+3}|}{|(i\epsilon_{12}^{1/2})^3\quad (i\overline{\epsilon}_{12}^{1/2})^3\quad (-i\epsilon_{12}^{1/2})^3\quad (-i\overline{\epsilon}_{12}^{1/2})^3|} \\ & \quad \frac{|\epsilon_{12}^{2s+1}\quad \overline{\epsilon}_{12}^{2s+1}|}{|\epsilon_{12}\quad \overline{\epsilon}_{12}|} \quad \Delta\overline{\Delta} \quad d\phi_{12} \end{aligned} \tag{5.122}$$

Taking the complex conjugate of the $SU(4)$ character permutes columns in the matrices but any change of sign in the numerator is cancelled by a similar factor from the denominator. We will rearrange this equation into a form where we can use character orthogonality to evaluate the integrals. Let

$$e^{i\psi} = \eta = \epsilon_{12}^{1/2}$$

Changing variables in the integral

$$\begin{aligned} A_{12} &= \frac{2}{\Omega_{12}} \int_0^{\pi} \frac{|(i\eta)^{f_1+3}\ (i\overline{\eta})^{f_1+3}\ (-i\eta)^{f_1+3}\ (-i\overline{\eta})^{f_1+3}|}{|(i\eta)^3\quad (i\overline{\eta})^3\quad (-i\eta)^3\quad (-i\overline{\eta})^3|} \\ & \quad \frac{|\eta^{2(2s+1)}\ \overline{\eta}^{2(2s+1)}|}{|\eta^2\quad \overline{\eta}^2|} \quad (\Delta\overline{\Delta})' \quad d\psi \end{aligned} \tag{5.123}$$

The integral (5.122) with respect to ϕ is periodic with period 2π so the integral with respect to ψ is periodic with period π. Using this the integral (5.123) can be returned to an integral over $SU(2)$. Applying the Littlewood-Richardson factorisation (5.118) and some algebra we obtain

$$\begin{aligned} A_{12} \quad &= \quad \frac{i^{|\mathbf{f}|}}{\Omega_{12}} \int_0^{2\pi} \sum_{\boldsymbol{\alpha\beta}} Y^{\boldsymbol{f}}_{\boldsymbol{\alpha\beta}} \frac{|\eta^{\alpha_1+1}\;\overline{\eta}^{\alpha_1+1}|}{|\eta \quad \overline{\eta}|} \frac{|(-\eta)^{\beta_1+1}\;(-\overline{\eta})^{\beta_1+1}|}{|-\eta \quad -\overline{\eta}|} \\ &\left(\frac{|\eta^{2(2s+1)+1}\;\overline{\eta}^{2(2s+1)+1}|}{|\eta \quad \overline{\eta}|} + \frac{|\eta^{4s+1}\;\overline{\eta}^{4s+1}|}{|\eta \quad \overline{\eta}|} \right) \quad \Delta\overline{\Delta}\, d\psi \end{aligned} \tag{5.124}$$

This is a sum of integrals over $SU(2)$ of the product of three characters of $SU(2)$. Solving such an integral is equivalent to decomposing the tensor product of two irreducible representations of $SU(2)$, the solutions of which are the Clebsch-Gordan coefficients.

$$C_{\alpha\beta\gamma} = \frac{1}{\Omega} \int_0^{2\pi} \overline{\chi}^{\alpha}_{SU(2)}(\epsilon) \overline{\chi}^{\beta}_{SU(2)}(\epsilon) \chi^{\gamma}_{SU(2)}(\epsilon) \quad \Delta\overline{\Delta} d\phi \tag{5.125}$$

The Clebsch-Gordan coefficients for $SU(2)$ are given by

$$C_{\alpha\beta\gamma} = \begin{cases} 1 & \text{if } |\alpha - \beta| \leq \gamma \leq \alpha + \beta \\ 0 & \text{otherwise} \end{cases} \tag{5.126}$$

To apply this to the integral for A_{12} we recall that the representation $\boldsymbol{\alpha}$ of $SU(2)$ is equivalent to the representation $(\alpha_1 - \alpha_2, 0)$ and define the integer α to be $\alpha_1 - \alpha_2$. We also know from the Weyl character formula that the characters of $SU(2)$ are real. Solving for A_{12}

$$A_{12} = i^{|\mathbf{f}|} \sum_{\boldsymbol{\alpha\beta}} Y^{\mathbf{f}}_{\boldsymbol{\alpha\beta}} (-1)^{|\boldsymbol{\beta}|} \left[C_{\alpha\,\beta\,(4s+2)} + C_{\alpha\,\beta\,4s} \right] \tag{5.127}$$

We have now evaluated the second integral in the formula (5.111) using the Littlewood-Richardson and Clebsch-Gordan coefficients.

Before combining the results for A_{12} and A_I we can state another property of A_{12}. An irreducible representation $B^{s\lambda}(\mathcal{H})$ restricted to the subgroup of $\mathcal{H}$ connected to the identity is a tensor product of the representations R^s of $SU(2)$. This is an irreducible representation of $SU(2) \times SU(2)$. If a representation of $SU(4)$ contains no representation of this subgroup, ie $A_I = 0$, then we also know it can not contain the corresponding representation of $\mathcal{H}$. Therefore A_{12} is zero if A_I is zero.

The rules for multiplying Young tableau determine the coefficient $Y^{\mathbf{f}}_{\mathbf{ss}}$. The number of boxes in a tableau $\mathbf{f}$ which is the product of two tableau $\mathbf{s}$ is twice the number of boxes in $\mathbf{s}$. So if A_I is nonzero

$$|\mathbf{f}| = 2(s_1 + s_2) = 2(2s + 2s_2) \tag{5.128}$$

where s_2 is an integer but s can be half integer. Using this we can replace the phase $i^{|\mathbf{f}|}$ in (5.127) with $(-1)^{2s}$.

5.6.4 The number of physical representations of $\mathcal{H}$ in $SU(4)$

Substituting the expressions for A_I and A_{12} into equation (5.111) for the number of physical representations in the decomposition of the representation $\mathbf{f}$ of $SU(4)$ we find that

$$N^{\mathbf{f}}_{s\lambda} = \frac{1}{2}Y^{\mathbf{f}}_{\mathbf{ss}} + \chi^{\lambda}_{S_2}((12))\frac{(-1)^{2s}}{2}\sum_{\boldsymbol{\alpha\beta}} Y^{\mathbf{f}}_{\boldsymbol{\alpha\beta}}\,(-1)^{|\boldsymbol{\beta}|}\,[C_{\alpha\,\beta\,(4s+2)} + C_{\alpha\,\beta\,4s}] \tag{5.129}$$

This gives the number of physical irreducible representations $B_{s\lambda}(\mathcal{H})$ when the representation of $SU(4)$ is restricted to $\mathcal{H}$.

λ is a partition of two and if we add the results for the two possible values of $\chi^{\lambda}_{S_2}((12))$, which are ± 1, we find that

$$N^{\mathbf{f}}_{s(2)} + N^{\mathbf{f}}_{s(1,1)} = Y^{\mathbf{f}}_{\mathbf{ss}} \tag{5.130}$$

The number of representations of $\mathcal{H}$ with spin s from which we can select spin vectors $|M\rangle$ to use in the construction is given by the number of copies of the tableau $\mathbf{f}$ found when two identical tableau labelling a representation of $SU(2)$ with spin s are multiplied. This agrees with the result for the number of multiplets with spin s and zero weight with respect to E_z which we obtained directly in chapter 4.

From the results in chapter 4 we also expect that approximately half the multiplets available to the construction will transform according to each irreducible representation of S_2. For this to agree with equation (5.129) A_{12} should be zero when A_I is even and ± 1 for A_I odd. While we have not been able to establish this from these results we show that this is plausible. If we consider a choice of

$\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$ and $\boldsymbol{\beta} = (\beta_1, \beta_2)$ for which $Y^{\mathbf{f}}_{\boldsymbol{\alpha}\boldsymbol{\beta}}$ is not zero then changing $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ slightly doesn't change $Y^{\mathbf{f}}_{\boldsymbol{\alpha}\boldsymbol{\beta}}$. For example

$$Y^{\mathbf{f}}_{(\alpha_1,\alpha_2)(\beta_1,\beta_2)} = Y^{\mathbf{f}}_{(\alpha_1-1,\alpha_2)(\beta_1+1,\beta_2)} \tag{5.131}$$

The sign $(-1)^{|\boldsymbol{\beta}|}$ multiplying these two terms in the sum is different. Consequently the terms cancel. Unfortunately this argument can not be applied to all shapes of tableau. Using this procedure we see that we expect most terms in A_{12} to cancel and we will be left with a small integer although restricting this to 0 or ± 1 has not been achieved.

In section 4.6 we evaluated the exchange signs of spin multiplets in the low dimensional irreducible representations of $SU(4)$ numerically. For these representations labelled by tableau with up to six boxes we can also determine the number of multiplets with each exchange sign using (5.129). The analytic results from the character decomposition agree with those computed directly.

5.7 The decomposition of $SU(6)$

Before tackling the general case it will be useful to see how the techniques introduced to solve the integrals in the decomposition of $SU(4)$ are modified when the symmetric group is less trivial. If we take n to be three we are dealing with representations of the classical groups $SU(6)$ and S_3. S_3 is a group of six elements in three classes, the identity, three two cycles, and two three cycles. The formula for the character decomposition will now involve a sum over these three classes. Rewriting equations (5.70) and (5.71) for the integral of the characters over $\mathcal{H}$ when $n = 3$

$$N^{\mathbf{f}}_{s\lambda} = \frac{1}{6} A_I + \chi^{\lambda}_{S_3}((12)) \frac{3}{6} A_{12} + \chi^{\lambda}_{S_3}((123)) \frac{2}{6} A_{123} \tag{5.132}$$

$$
\begin{aligned}
A_I &= \frac{1}{\Omega_I}\int_0^{2\pi}\cdots\int_0^{2\pi} \Delta\overline{\Delta}\, d\phi_1 d\phi_2 d\phi_3 d\theta_1 d\theta_2 \\
&\quad \overline{X}^{\mathbf{f}}_{SU(6)}(I,\boldsymbol{\epsilon}_I,\boldsymbol{\theta}_I)\,\chi^s_{\mathcal{H}}(I,\boldsymbol{\epsilon}_I) && (5.133)\\
A_{12} &= \frac{1}{\Omega_{12}}\int_0^{2\pi}\cdots\int_0^{2\pi} \Delta\overline{\Delta}\, d\phi_{12} d\phi_3 d\theta_{12} \\
&\quad \overline{X}^{\mathbf{f}}_{SU(6)}((12),\boldsymbol{\epsilon}_{12},\boldsymbol{\theta}_{12})\,\chi^s_{\mathcal{H}}((12),\boldsymbol{\epsilon}_{12}) && (5.134)\\
A_{123} &= \frac{1}{\Omega_{123}}\int_0^{2\pi} \overline{X}^{\mathbf{f}}_{SU(6)}((123),\boldsymbol{\epsilon}_{123})\,\chi^s_{\mathcal{H}}((123),\boldsymbol{\epsilon}_{123})\,\Delta\overline{\Delta}\, d\phi_{123} && (5.135)
\end{aligned}
$$

We will solve the integrals for the A's in turn, factorising the determinants until we reach products of $SU(2)$ characters.

5.7.1 Evaluation of A_I

For elements of $\mathcal{H}$ connected to the identity the character of the representation of $SU(6)$ is

$$
\frac{|(e^{i\theta_1}\epsilon_1)^{f_1+5}\;(e^{i\theta_1}\overline{\epsilon}_1)^{f_1+5}\;(e^{i\theta_2}\epsilon_2)^{f_1+5}\;(e^{i\theta_2}\overline{\epsilon}_2)^{f_1+5}\;(e^{i\theta_3}\epsilon_3)^{f_1+5}\;(e^{i\theta_3}\overline{\epsilon}_3)^{f_1+5}|}{|(e^{i\theta_1}\epsilon_1)^5\quad(e^{i\theta_1}\overline{\epsilon}_1)^5\quad(e^{i\theta_2}\epsilon_2)^5\quad(e^{i\theta_2}\overline{\epsilon}_2)^5\quad(e^{i\theta_3}\epsilon_3)^5\quad(e^{i\theta_3}\overline{\epsilon}_3)^5\,|}
$$

This is found by substituting the eigenvalues from (5.110) into the Weyl character formula for $SU(6)$. We will apply the Littlewood-Richardson factorisation (5.118) twice to the character of $SU(6)$. Splitting the character first into the sum of the products of characters of $SU(4)$ and $SU(2)$. Then factorising the character of $SU(4)$ in the product of two characters of $SU(2)$. Substituting the characters into equation (5.133) for A_I

$$
\begin{aligned}
A_I &= \frac{1}{\Omega_I}\int_0^{2\pi}\cdots\int_0^{2\pi}\sum_{\boldsymbol{\alpha\beta\gamma}} Y^{\mathbf{f}}_{\boldsymbol{\alpha\beta\gamma}}\,\Delta\overline{\Delta}\, d\phi_1 d\phi_2 d\phi_3 d\theta_1 d\theta_2 \\
&\quad e^{i(|\boldsymbol{\alpha}|\theta_1+|\boldsymbol{\beta}|\theta_2+|\boldsymbol{\gamma}|\theta_3)}\frac{|\epsilon_1^{\alpha_1+1}\;\overline{\epsilon}_1^{\alpha_1+1}|}{|\epsilon_1\quad\overline{\epsilon}_1|}\frac{|\epsilon_2^{\beta_1+1}\;\overline{\epsilon}_2^{\beta_2+1}|}{|\epsilon_2\quad\overline{\epsilon}_2|}\frac{|\epsilon_3^{\gamma_1+1}\;\overline{\epsilon}_3^{\gamma_1+1}|}{|\epsilon_3\quad\overline{\epsilon}_3|} \\
&\quad \frac{|\epsilon_1^{2s+1}\;\overline{\epsilon}_1^{2s+1}|}{|\epsilon_1\quad\overline{\epsilon}_1|}\frac{|\epsilon_2^{2s+1}\;\overline{\epsilon}_2^{2s+1}|}{|\epsilon_2\quad\overline{\epsilon}_2|}\frac{|\epsilon_3^{2s+1}\;\overline{\epsilon}_3^{2s+1}|}{|\epsilon_3\quad\overline{\epsilon}_3|} && (5.136)
\end{aligned}
$$

The Littlewood-Richardson coefficients, $Y^{\mathbf{f}}_{\boldsymbol{\alpha\beta\gamma}}$, which appear when the factorisation is applied twice are the number of tableau $\mathbf{f}$ in the product of the three tableau $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$.

To solve the phase integrals we use the condition, $e^{i\sum\theta_j} = 1$, to eliminate θ_3. Then for A_I to be nonzero

$$|\boldsymbol{\alpha}| = |\boldsymbol{\beta}| = |\boldsymbol{\gamma}| \tag{5.137}$$

The three ϕ-integrals are each products of two irreducible characters of $SU(2)$ integrated over $SU(2)$. From character orthogonality either both characters correspond to the same irreducible representation of $SU(2)$ or the integral is zero. We have the second condition

$$\alpha = \beta = \gamma = 2s \tag{5.138}$$

where α is defined as $(\alpha_1 - \alpha_2)$. The volume Ω_I is cancelled when the integral is nonzero. Combining the two conditions we find that

$$A_I = Y^{\mathbf{f}}_{\mathbf{sss}} \tag{5.139}$$

$\mathbf{s}$ is defined as previously to be a vector (s_1, s_2) where $s_1 - s_2 = 2s$. It is clear that this result for A_I will generalise to any value of n. A_I is the number of tableau $\mathbf{f}$ in the product of n identical tableau with spin s.

5.7.2 Evaluation of A_{12}

The character of $SU(6)$ in this case will be similar to that used in the integral for A_I however the eigenvalues will be replaced by powers of $(ie^{i\theta_{12}/2}\epsilon_{12}^{1/2})$, $(-ie^{i\theta_{12}/2}\epsilon_{12}^{1/2})$, $(e^{-i\theta_{12}}\epsilon_3)$ and the equivalent terms with $\bar{\epsilon}$. These eigenvalues are the solutions of (5.110) for a single two cycle. We first apply the Littlewood-Richardson factorisation once to separate the terms involving ϵ_3. We rewrite (5.134) as

$$\begin{aligned} A_{12} &= \frac{1}{\Omega_{12}} \int_0^{2\pi} \cdots \int_0^{2\pi} \sum_{\boldsymbol{\alpha\beta}} Y^{\mathbf{f}}_{\boldsymbol{\alpha\beta}} \, \Delta\overline{\Delta} \, d\phi_{12} d\phi_3 d\theta_{12} \\ & \quad e^{i(|\boldsymbol{\alpha}|\theta_{12}/2 - |\boldsymbol{\beta}|\theta_3)} \frac{|(i\epsilon_{12}^{1/2})^{\alpha_1+3} \; (i\bar{\epsilon}_{12}^{1/2})^{\alpha_1+3} \; (-i\epsilon_{12}^{1/2})^{\alpha_1+3} \; (-i\bar{\epsilon}_{12}^{1/2})^{\alpha_1+3}|}{|(i\epsilon_{12}^{1/2})^3 \quad (i\bar{\epsilon}_{12}^{1/2})^3 \quad (-i\epsilon_{12}^{1/2})^3 \quad (-i\bar{\epsilon}_{12}^{1/2})^3|} \\ & \quad \frac{|\epsilon_3^{\beta_1+1} \; \bar{\epsilon}_3^{\beta_2+1}|}{|\epsilon_3 \quad \bar{\epsilon}_3|} \frac{|\epsilon_{12}^{2s+1} \; \bar{\epsilon}_{12}^{2s+1}|}{|\epsilon_{12} \quad \bar{\epsilon}_{12}|} \frac{|\epsilon_3^{2s+1} \; \bar{\epsilon}_3^{2s+1}|}{|\epsilon_3 \quad \bar{\epsilon}_3|} \end{aligned} \tag{5.140}$$

We can now evaluate the phase integral. A_{12} is nonzero only if

$$|\boldsymbol{\alpha}| = 2|\boldsymbol{\beta}| \tag{5.141}$$

The integral with respect to ϕ_3 is the integral of two irreducible characters of $SU(2)$ from which we find the condition

$$\beta = 2s \tag{5.142}$$

for A_{12} nonzero. The integral for ϕ_{12} is the same as the integral for A_{12} in $SU(4)$ where the representation $\mathbf{f}$ is replace by $\boldsymbol{\alpha}$. Combining these results we have determined A_{12}

$$A_{12} = (-1)^{2s} \sum_{\boldsymbol{\alpha}} Y^{\mathbf{f}}_{\boldsymbol{\alpha}\mathbf{g}} \sum_{\boldsymbol{\gamma}\boldsymbol{\delta}} Y^{\boldsymbol{\alpha}}_{\boldsymbol{\gamma}\boldsymbol{\delta}} (-1)^{|\boldsymbol{\delta}|} \left[C_{\gamma\,\delta\,(4s+2)} + C_{\gamma\,\delta\,4s}\right] \tag{5.143}$$

We see that the solution still involves two sums over the coefficients $Y^{\boldsymbol{\gamma}}_{\boldsymbol{\alpha}\boldsymbol{\beta}}$. Effectively the solution of A_{12} when $n = 3$ requires knowledge of all the solutions of A_{12} from $n = 2$.

5.7.3 Evaluation of A_{123}

A_{123} is defined in equation (5.135) into which we substitute the Weyl characters of $SU(6)$ and $\mathcal{H}$. The eigenvalues of the elements of $\mathcal{H}$ connected to a three cycle in S_3 are $\epsilon_{123}^{1/3}$ and $\bar{\epsilon}_{123}^{1/3}$ multiplied by each of the cube roots of unity. We simplify the formulae for A_{123} by changing the variable so that

$$\epsilon_{123}^{1/3} = \eta = e^{i\psi} \tag{5.144}$$

By applying the the Littlewood-Richardson factorisation twice we reduce the character of $SU(6)$ into a product

$$\begin{aligned} A_{123} \quad = \quad & \frac{1}{\Omega_{123}} \int_0^{2\pi} \sum_{\boldsymbol{\alpha}\boldsymbol{\beta}\boldsymbol{\gamma}} Y^{\mathbf{f}}_{\boldsymbol{\alpha}\boldsymbol{\beta}\boldsymbol{\gamma}} \, (\Delta\overline{\Delta})' \, d\psi \, e^{i\frac{2\pi}{3}(|\boldsymbol{\alpha}|-|\boldsymbol{\gamma}|)} \quad \frac{|\eta^{\alpha_1+1}\,\overline{\eta}^{\alpha_1+1}|}{|\eta \quad \overline{\eta}|} \\ & \frac{|\eta^{\beta_1+1}\,\overline{\eta}^{\beta_1+1}|}{|\eta \quad \overline{\eta}|} \, \frac{|\eta^{\gamma_1+1}\,\overline{\eta}^{\gamma_1+1}|}{|\eta \quad \overline{\eta}|} \, \frac{|\eta^{3(2s+1)}\,\overline{\eta}^{3(2s+1)}|}{|\eta^3 \quad \overline{\eta}^3|} \end{aligned} \tag{5.145}$$

The expresion can now be written as a product of $SU(2)$ characters

$$\begin{aligned} A_{123} \quad = \quad & \frac{1}{\Omega_{123}} \int_0^{2\pi} \sum_{\boldsymbol{\alpha}\boldsymbol{\beta}\boldsymbol{\gamma}} Y^{\mathbf{f}}_{\boldsymbol{\alpha}\boldsymbol{\beta}\boldsymbol{\gamma}} \, \Delta\overline{\Delta} \, d\psi \quad e^{i\frac{2\pi}{3}(|\boldsymbol{\alpha}|-|\boldsymbol{\gamma}|)} \\ & \left(\frac{|\eta^{(6s+4)+1}\,\overline{\eta}^{(6s+4)+1}|}{|\eta \quad \overline{\eta}|} + \frac{|\eta^{(6s+2)+1}\,\overline{\eta}^{(6s+2)+1}|}{|\eta \quad \overline{\eta}|} + \frac{|\eta^{(6s)+1}\,\overline{\eta}^{(6s)+1}|}{|\eta \quad \overline{\eta}|} \right) \\ & \frac{|\eta^{\alpha_1+1}\,\overline{\eta}^{\alpha_1+1}|}{|\eta \quad \overline{\eta}|} \, \frac{|\eta^{\beta_1+1}\,\overline{\eta}^{\beta_1+1}|}{|\eta \quad \overline{\eta}|} \, \frac{|\eta^{\gamma_1+1}\,\overline{\eta}^{\gamma_1+1}|}{|\eta \quad \overline{\eta}|} \end{aligned} \tag{5.146}$$

This has reduced the expression for A_{123} to a sum of three integrals over $SU(2)$ of the product of four irreducible characters of $SU(2)$. The integral of a product of four characters of $SU(2)$ is an extension of the Clebsch-Gordan coefficients the Racah coefficients, see [48]. They are functions of the four integers which label the irreducible representations of $SU(2)$ and in correspondence with the Clebsch-Gordan coefficients we label them $C_{\alpha\beta\gamma\delta}$. Using this notation the term A_{123} is

$$A_{123} = \sum_{\boldsymbol{\alpha\beta\gamma}} Y^{\mathbf{f}}_{\boldsymbol{\alpha\beta\gamma}} \, e^{i\frac{2\pi}{3}(|\boldsymbol{\alpha}|-|\boldsymbol{\gamma}|)} \left[C_{\alpha\beta\gamma(6s+4)} + C_{\alpha\beta\gamma(6s+2)} + C_{\alpha\beta\gamma(6s)}\right] \tag{5.147}$$

This is a sum of generalised Littlewood-Richardson coefficients multiplying generalised Clebsch-Gordan coefficients.

From the group theory we know that A_{123} should be an integer but the expression contains the phase $e^{i\frac{2\pi}{3}(|\boldsymbol{\alpha}|-|\boldsymbol{\gamma}|)}$. If a term in the sum

$$Y^{\mathbf{f}}_{\boldsymbol{\alpha\beta\gamma}}[C_{\alpha\beta\gamma(6s+4)} + C_{\alpha\beta\gamma(6s+2)} + C_{\alpha\beta\gamma(6s)}]$$

is nonzero for one choice of $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ then it will be the same for any permutation of $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$. In particular exchanging $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$ the term will have the conjugate phase which ensures that the sum over all $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$ is an integer.

5.7.4 Physical representations of $\mathcal{H}$ in $SU(6)$

The solutions of equations (5.133) to (5.135) which define A_I, A_{12} and A_{123} combined with the irreducible characters of S_3 provide an analytic decomposition of $SU(6)$ into those representations physically relevant to the construction of a position dependent spin basis.

To find the multiplicity of any particular physical representation of $\mathcal{H}$ the solutions for A_I, A_{12} and A_{123} which depend on s are substituted into the sum over the permutation group (5.132) where the representation λ of the permutation group enters.

$$N^{\mathbf{f}}_{s\lambda} = \frac{1}{6}\chi^{\lambda}_{S_3}(I)A_I + \frac{3}{6}\chi^{\lambda}_{S_3}((12))A_{12} + \frac{2}{6}\chi^{\lambda}_{S_3}((123))A_{123} \tag{5.148}$$

The characters of the irreducible representations of S_3 which appear in (5.148) are recorded in figure 5.1.

	I	(12)	(123)
$\chi_{S_3}^{(3)}$	1	1	1
$\chi_{S_3}^{(2,1)}$	2	0	-1
$\chi_{S_3}^{(1,1,1)}$	1	-1	1

Figure 5.1: Irreducible characters of S_3

The total number of physical representations with spin s in the given representation of $SU(6)$ is found by summing $N^{\mathbf{f}}_{s\lambda}$ over the three irreducible representations λ of S_n.

$$N^{\mathbf{f}}_{s} = \frac{2}{3}A_I + \frac{1}{3}A_{123} \tag{5.149}$$

We see that in this case the simple solution for A_I does not determine the total number of physical representations as it did for representations of $SU(4)$. Unfortunately not only is the solution for $SU(6)$ more complex but the factors that are more difficult to evaluate, A_{12} and A_{123}, are more significant.

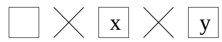
5.7.5 Example: The $(2,1)$ representation of $SU(6)$ contains a spin-$1/2$ multiplet which exhibits parastatistics.

With this example we can see how equation (5.148) is used to evaluate the exchange sign. The calculation also provides an explicit case of parastatistics in a representation of $SU(6)$. For three particles parastatistics corresponds to vectors transforming according to the two dimensional irreducible representation of S_3. The character of this representation is recorded in the second row of figure 5.1. In this example both $\mathbf{f}$ and λ are labelled by the partition $(2,1)$. From (5.148)

$$N^{(2,1)}_{1/2\,(2,1)} = \frac{2}{6}A_I - \frac{2}{6}A_{123} \tag{5.150}$$

As $\chi_{S_3}^{(2,1)}((12))$ is zero we avoid needing to evaluate A_{12}.

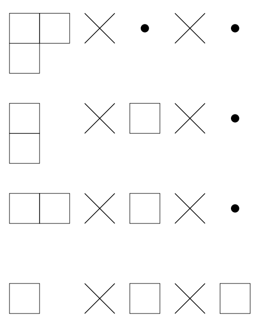
$A_I = Y^{(2,1)}_{\mathbf{sss}}$ which is the number of ways of multiplying three tableau, consisting of a single box for spin-1/2, and obtaining the tableau $(2,1)$. This is the tableau multiplication



where the labels x and y are used to distinguish the tableau. From the rules for tableau multiplication, section 2.9.1, there are two distinct results with shape $(2,1)$

Consequently A_I is 2.

To evaluate A_{123} using equation (5.147) we must consider all possible ways of multiplying three tableau and obtaining $(2,1)$. As the representation is simple these can be written out in full.



The dot denotes a tableau with no boxes. Except for the last case which we evaluated for A_I the coefficients $Y^{(2,1)}_{\boldsymbol{\alpha\beta\gamma}}$ are unity for each of these multiplication schemes. In each case we must also evaluate the sum of coefficients

$$\widehat{C} = C_{\alpha\beta\gamma(6s+4)} + C_{\alpha\beta\gamma(6s+2)} + C_{\alpha\beta\gamma(6s)} \tag{5.151}$$

for $s = 1/2$. This can be done by applying the Clebsch-Gordan formula twice.

$$(2,1) \times 0 \times 0 \qquad \widehat{C} = 0 \tag{5.152}$$

$$(1,1) \times (1) \times 0 \qquad \widehat{C} = 0 \tag{5.153}$$

$$(2) \times (1) \times 0 \qquad \widehat{C} = 1 \tag{5.154}$$

$$(1) \times (1) \times (1) \qquad \widehat{C} = 1 \tag{5.155}$$

For the two non zero cases we must also consider the possible phase factors

$$e^{i\frac{2\pi}{3}(|\boldsymbol{\alpha}|-|\boldsymbol{\gamma}|)}$$

that can occur. For $(1)\times(1)\times(1)$ as the three tableau are identical there is only a single contribution of this type and the phase factor is unity. Using $Y^{(2,1)}_{(1)(1)(1)}=2$ the contribution to A_{123} from the tableau multiplication of this type is

$$\widehat{C}\,Y^{(2,1)}_{(1)(1)(1)}=2 \tag{5.156}$$

For $(2)\times(1)\times 0$ the tableau are all different and there are 6 such multiplications depending on which tableau is assigned to which particle. In three cases the phase is $e^{i\frac{2\pi}{3}}$ and in three it is $e^{-i\frac{2\pi}{3}}$. The contribution to A_{123} from the tableau multiplications of this form is

$$(3e^{i\frac{2\pi}{3}}+3e^{-i\frac{2\pi}{3}})(\widehat{C}\,Y^{(2,1)}_{(2)(1)0})=-3 \tag{5.157}$$

Summing the two non zero contributions $A_{123}=-1$.

We can substitute the two results for A_I and A_{123} into equation (5.150) to obtain the final result

$$N^{(2,1)}_{1/2\,(2,1)}=1 \tag{5.158}$$

The representation $(2,1)$ of $SU(6)$ contains a single spin-1/2 subspace transforming according to the two dimensional representation of S_3. If we also use our results for A_I and A_{123} in equation (5.149) we see that this is the only physical representation of $\mathcal{H}$ which is contained in this irreducible representation of $SU(6)$. Wavefunctions on a position-dependent spin basis constructed from this representation will exhibit parastatistics.

While this result is interesting we should not be surprised by it. There is a simpler line of reasoning which leads to the same conclusion. If we notice that the representation of $SU(6)$ is labeled by the same tableau $(2,1)$ which labels the two dimensional representation of S_3 we can see that vectors generated by these symmetry conditions must necessarily transform according to the given irreducible representation of the symmetric group. While this can not be applied to larger more complex tableau it does at least show that for spin-1/2 there exists a representation of $SU(2n)$ where the position dependent basis transforms according to every possible representation of S_n. All types of parastatistics can be exhibited in some position-dependent spin basis.

5.8 The decomposition of $SU(2n)$

Having tackled a more typical example in the decomposition of $SU(6)$ we can apply the same techniques to evaluate the coefficients A_κ in the decomposition of an irreducible representation of $SU(2n)$ into physical representations of $\mathcal{H}$.

The sum over $\mathcal{H}$ of a product of characters of $\mathcal{H}$ was defined in equation (5.70). Inserting the characters of the physical representations of $\mathcal{H}$ and the irreducible representation of $SU(2n)$ into this formula we have an equation for the number of representations $B^{s\lambda}(\mathcal{H})$ in the decomposition of the representation of $SU(2n)$.

$$N^{\mathbf{f}}_{s\lambda} = \frac{1}{\Omega_{S_n}} \sum_{\kappa} \Omega_{\kappa S_n} \chi^{\lambda}_{S_n}(\kappa) A_\kappa \tag{5.159}$$

where

$$\begin{aligned} A_\kappa = \frac{1}{\Omega_{\kappa\mathcal{H}}} \int_0^{2\pi} \cdots \int_0^{2\pi} \Delta\overline{\Delta} d\theta_1 \ldots d\theta_{r-1} d\phi_1 \ldots d\phi_r \\ \overline{X}^{\mathbf{f}}_{SU(2n)}(\kappa, \boldsymbol{\theta}_\kappa, \boldsymbol{\epsilon}_\kappa) \chi^{s}_{\mathcal{H}}(\kappa, \boldsymbol{\epsilon}_\kappa) \end{aligned} \tag{5.160}$$

κ is a partition of n labelling a class of S_n and r is the number of cycles in the class κ. The irreducible characters of S_n are known, as are the number of elements in the classes of the symmetric group. Therefore the problem is to determine A_κ for a general class κ. With this $N^{\mathbf{f}}_{s\lambda}$ can be written as a sum of known coefficients.

5.8.1 Evaluation of A_κ

$\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_r)$ is a partition of n. From equation (5.110) the eigenvalues of elements of $\mathcal{H}$ connected to an element of S_n in the class κ are of the form

$$e^{i2\pi p/\kappa_j}\, e^{i\theta_j/\kappa_j}\, \epsilon_j^{1/\kappa_j} = \varepsilon_{jp} \quad \text{or} \quad e^{i2\pi p/\kappa_j}\, e^{i\omega_j/\kappa_j}\, \overline{\epsilon}_j^{1/\kappa_j} = \tilde{\varepsilon}_{jp} \tag{5.161}$$

where p is an integer and ϵ_j an eigenvalue of $SU(2)$. There are $2\kappa_j$ eigenvalues for each cycle j.

Using the symbols ε_{jp} and $\tilde{\varepsilon}_{jp}$ for the eigenvalues we can write the Weyl character formula for the representation of $SU(2n)$.

$$\frac{|\varepsilon_{11}^{f_1+(2n-1)} \ldots \tilde{\varepsilon}_{1\kappa_1}^{f_1+(2n-1)} \varepsilon_{21}^{f_1+(2n-1)} \ldots \tilde{\varepsilon}_{2\kappa_2}^{f_1+(2n-1)} \ldots \varepsilon_{r1}^{f_1+(2n-1)} \ldots \tilde{\varepsilon}_{r\kappa_r}^{f_1+(2n-1)}|}{|\varepsilon_{11}^{(2n-1)} \ldots \tilde{\varepsilon}_{1\kappa_1}^{(2n-1)} \varepsilon_{21}^{(2n-1)} \ldots \tilde{\varepsilon}_{2\kappa_2}^{(2n-1)} \ldots \varepsilon_{r1}^{(2n-1)} \ldots \tilde{\varepsilon}_{r\kappa_r}^{(2n-1)}|}$$

With the Littlewood-Richardson theorem we split this ratio of $2n \times 2n$ determinants into a sum of products of determinants of size $2\kappa_j \times 2\kappa_j$ one for each cycle in the class κ. Each of these irreducible characters of $SU(2\kappa_j)$ is labelled by a vector of integers $\boldsymbol{\beta}^j$.

$$X^{\mathbf{f}}_{SU(2n)}(\kappa, \boldsymbol{\theta}_\kappa, \boldsymbol{\epsilon}_\kappa) = \sum_{\boldsymbol{\beta}} Y^{\mathbf{f}}_{\boldsymbol{\beta}^1\ldots\boldsymbol{\beta}^r} \prod_{\boldsymbol{\beta}^1\ldots\boldsymbol{\beta}^r} \frac{|\varepsilon_{j1}^{\beta_1^j+(2\kappa_j-1)} \ldots \tilde{\varepsilon}_{j\kappa_j}^{\beta_1^j+(2\kappa_j-1)}|}{|\varepsilon_{j1}^{(2\kappa_j-1)} \ldots \tilde{\varepsilon}_{j\kappa_j}^{(2\kappa_j-1)}|} \tag{5.162}$$

Using this factorisation we can rewrite equation (5.160). The phase $e^{-i\theta_j/\kappa_j}$ can be taken out of the determinants labelled by $\boldsymbol{\beta}^j$. We obtain the integral

$$A_\kappa = \frac{1}{\Omega_{\kappa\mathcal{H}}} \int_0^{2\pi} \cdots \int_0^{2\pi} \sum_{\boldsymbol{\beta}} e^{-i\sum_j \frac{|\boldsymbol{\beta}^j|}{\kappa_j}\theta_j} Y^{\mathbf{f}}_{\boldsymbol{\beta}^1\ldots\boldsymbol{\beta}^r} \prod_{\boldsymbol{\beta}^1\ldots\boldsymbol{\beta}^r} A_{\boldsymbol{\beta}^j}\, d\theta_1 \ldots d\theta_{r-1} \tag{5.163}$$

where $A_{\boldsymbol{\beta}}$ is an integral over one of the eigenvalues ϵ of $SU(2)$ defined by

$$\begin{aligned} A_{\boldsymbol{\beta}^j} \quad &= \quad \int_0^{2\pi} \Delta\overline{\Delta}\, d\phi_j \frac{|(e^{i2\pi/\kappa_j}\epsilon_j^{1/\kappa_j})^{\beta_1^j+(2\kappa_j-1)} \ldots (e^{i2\pi\kappa_j/\kappa_j}\overline{\epsilon}_j^{1/\kappa_j})^{\beta_1^j+(2\kappa_j-1)}|}{|(e^{i2\pi/\kappa_j}\epsilon_j^{1/\kappa_j})^{(2\kappa_j-1)} \ldots (e^{i2\pi\kappa_j/\kappa_j}\overline{\epsilon}_j^{1/\kappa_j})^{(2\kappa_j-1)}|} \\ &\quad \frac{|\epsilon_j^{2s+1}\ \overline{\epsilon}_j^{2s+1}|}{|\epsilon_j \quad \overline{\epsilon}_j|} \end{aligned} \tag{5.164}$$

Each of the κ_j'th roots of unity appear in the character labelled by $\boldsymbol{\beta}^j$. The character labelled by $\boldsymbol{\beta}^j$ is real as taking the complex conjugate permutes the columns in the determinant but a sign change in the numerator would be cancelled by the corresponding sign from the denominator.

Following this factorisation we will first solve the phase integrals in (5.163) then find a general solution of integrals of the form of (5.164). To solve the phase integrals we use the condition

$$e^{i\sum\theta_j} = \mathrm{sgn}(\kappa)$$

As all elements of S_n in the same class have the same cycle structure $\mathrm{sgn}(\rho)$ is a function of the classes κ of ρ in S_n. We use this to eliminate θ_r

$$\theta_r = -\theta_1 - \cdots - \theta_{r-1} \quad (+\pi \quad \text{if } \kappa \text{ odd})$$

Then solving the phase integrals either A_κ is zero or the factors $\boldsymbol{\beta}$ obey the condition

$$\frac{|\boldsymbol{\beta}^j|}{\kappa_j} = \frac{|\boldsymbol{\beta}^r|}{\kappa_r} \quad \text{for all } j \tag{5.165}$$

If A_κ is nonzero the $r-1$ phase integrals produce a factor of $(2\pi)^{r-1}$ which cancels the similar term in $\Omega_{\kappa\mathcal{H}}$ and a phase $(\mathrm{sgn}(\kappa))^{|\boldsymbol{\beta}^r|/\kappa_r}$.

To solve the integral over $SU(2)$ for $A_{\boldsymbol{\beta}^j}$ defined in (5.164) we will first change the variable to simplify the notation.

$$\epsilon^{1/\kappa_j} = \eta = e^{i\psi}$$

Substituting into the equation for $A_{\boldsymbol{\beta}^j}$

$$\begin{aligned} A_{\boldsymbol{\beta}^j} &= \kappa_j \int_0^{2\pi/\kappa_j} (\Delta\overline{\Delta})'\, d\psi\, \frac{|(e^{i2\pi/\kappa_j}\eta)^{\beta_1^j+(2\kappa_j-1)} \ldots (e^{i2\pi\kappa_j/\kappa_j}\overline{\eta})^{\beta_1^j+(2\kappa_j-1)}|}{|(e^{i2\pi/\kappa_j}\eta)^{(2\kappa_j-1)} \ldots (e^{i2\pi\kappa_j/\kappa_j}\overline{\eta})^{(2\kappa_j-1)}|} \\ &\quad \frac{|\eta^{\kappa_j(2s+1)}\,\overline{\eta}^{\kappa_j(2s+1)}|}{|\eta^{\kappa_j} \quad \overline{\eta}^{\kappa_j}|} \end{aligned} \tag{5.166}$$

$(\Delta\overline{\Delta})'$ is the function of η that results from substituting for ϵ in the volume element $\Delta\overline{\Delta}$. The integral is periodic with period $2\pi/\kappa_j$ and using this we return equation (5.166) to an integral over $SU(2)$. We apply the Littlewood-Richardson factorisation to separate the different roots of unity in the character $\boldsymbol{\beta}^j$.

$$\begin{aligned} A_{\boldsymbol{\beta}^j} &= \int_0^{2\pi} \sum_{\boldsymbol{\alpha}} Y^{\boldsymbol{\beta}^j}_{\boldsymbol{\alpha}^1\ldots\boldsymbol{\alpha}^{\kappa_j}} (\Delta\overline{\Delta})'\, d\psi \quad e^{i\frac{2\pi}{\kappa_j}\left(\sum_{l=1}^{\kappa_j} l|\boldsymbol{\alpha}^l|\right)} \\ &\quad \left(\prod_{l=1\ldots\kappa_j} \frac{|\eta^{\alpha_1^l+1}\,\overline{\eta}^{\alpha_1^l+1}|}{|\eta \quad \overline{\eta}|}\right) \frac{|\eta^{\kappa_j(2s+1)}\,\overline{\eta}^{\kappa_j(2s+1)}|}{|\eta^{\kappa_j} \quad \overline{\eta}^{\kappa_j}|} \end{aligned} \tag{5.167}$$

where $\boldsymbol{\alpha}^l = (\alpha_1^l, \alpha_2^l)$ label characters of $SU(2)$. Clearly this integral is very close to a product of $\kappa_j + 1$ characters of $SU(2)$ however to reduce the integral to a sum of known coefficients we must factorise $(\Delta\overline{\Delta})'$. We cancel Δ' and $|\eta^{\kappa_j}\,\overline{\eta}^{\kappa_j}|$ and multiply top and bottom by Δ. Factorising $\overline{\Delta}'$ we find that

$$\overline{\Delta}' = (\overline{\eta}^{\kappa_j} - \eta^{\kappa_j}) = \left(\sum_{x=1}^{[\kappa_j/2]} (\overline{\eta}^{\kappa_j+1-2x} + \eta^{\kappa_j+1-2x}) \quad (+1 \text{ if } \kappa_j \text{ odd}) \right) \overline{\Delta}$$

where the sum is up to $[\kappa_j/2]$ the integer part of $\kappa_j/2$ and $\overline{\Delta} = (\overline{\eta} - \eta)$. Substituting this into equation (5.167) and incorporating the extra polynomial in η into the character determined by s we have reduced the equation for $A_{\boldsymbol{\beta}^j}$ to a product of characters of $SU(2)$.

$$\begin{aligned} A_{\boldsymbol{\beta}^j} \quad = \quad & \int_0^{2\pi} \sum_{\boldsymbol{\alpha}} Y^{\boldsymbol{\beta}^j}_{\boldsymbol{\alpha}^1 \ldots \boldsymbol{\alpha}^{\kappa_j}} \Delta\overline{\Delta}\, d\psi \quad e^{i\frac{2\pi}{\kappa_j}\left(\sum_{l=1}^{\kappa_j} l|\boldsymbol{\alpha}^l|\right)} \left(\prod_{l=1\ldots\kappa_j} \frac{|\eta^{\alpha_1^l+1}\;\overline{\eta}^{\alpha_1^l+1}|}{|\eta \quad \overline{\eta}|} \right) \\ & \left(\sum_{y=1}^{\kappa_j} \frac{|\eta^{\kappa_j(2s)+2(y-1)+1} \quad \overline{\eta}^{\kappa_j(2s)+2(y-1)+1}|}{|\eta \quad \overline{\eta}|} \right) \end{aligned} \tag{5.168}$$

This is the type of integral for which we can write down a solution using the generalised Clebsch-Gordan coefficients.

$$A_{\boldsymbol{\beta}^j} = \sum_{\boldsymbol{\alpha}} Y^{\boldsymbol{\beta}^j}_{\boldsymbol{\alpha}^1 \ldots \boldsymbol{\alpha}^{\kappa_j}}\; e^{i\frac{2\pi}{\kappa_j}\left(\sum_{l=1}^{\kappa_j} l|\boldsymbol{\alpha}^l|\right)} \sum_{y=1}^{\kappa_j} C_{\alpha^1 \ldots \alpha^{\kappa_j}\;(\kappa_j(2s)+2(y-1))} \tag{5.169}$$

In this case as previously α^y is defined to be $(\alpha_1^y - \alpha_2^y)$.

5.8.2 Results for the decomposition of $SU(2n)$

Collecting together the results for the decomposition of $SU(2n)$ the number of physical representations labelled by s and λ contained in a representation $\mathbf{f}$ of $SU(2n)$ is

$$N^{\mathbf{f}}_{s\lambda} = \frac{1}{\Omega_{S_n}} \sum_{\kappa} \Omega_{\kappa S_n} \chi^{\lambda}_{S_n}(\kappa) A_{\kappa} \tag{5.170}$$

where κ labels a class of S_n. The coefficients A_κ are

$$A_\kappa = \sum_{\boldsymbol{\beta}} (\text{sgn}(\kappa))^{|\boldsymbol{\beta}^r|/\kappa_r}\, Y^{\mathbf{f}}_{\boldsymbol{\beta}^1 \ldots \boldsymbol{\beta}^r} \prod_{j=1}^{r} A_{\boldsymbol{\beta}^j} \tag{5.171}$$

with the condition

$$|\boldsymbol{\beta}^j|/\kappa_j = |\boldsymbol{\beta}^r|/\kappa_r \text{ for all } j$$

and where $A_{\boldsymbol{\beta}^j}$ is

$$A_{\boldsymbol{\beta}^j} = \sum_{\boldsymbol{\alpha}} Y^{\boldsymbol{\beta}^j}_{\boldsymbol{\alpha}^1 \ldots \boldsymbol{\alpha}^{\kappa_j}} e^{i\frac{2\pi}{\kappa_j}\left(\sum_{l=1}^{\kappa_j} l|\boldsymbol{\alpha}^l|\right)} \sum_{y=1}^{\kappa_j} C_{\alpha^1 \ldots \alpha^{\kappa_j}\ (\kappa_j(2s)+2(y-1))} \tag{5.172}$$

These results give an analytic decomposition of the representation of $SU(2n)$ into physical representations of $\mathcal{H}$ in terms of the Littlewood-Richardson and Clebsch-Gordan coefficients and the characters of the irreducible representations of S_n.

While this solution of the decomposition problem is complete to evaluate $N^{\mathbf{f}}_{s\lambda}$ by hand we require both n and $|\mathbf{f}|$ to be small. This reduces the number of coefficients Y and C required to a manageable quantity which can be calculated. For larger values of n and $|\mathbf{f}|$ there do however exist algorithms for calculating the coefficients involved in the decomposition. The decomposition of any particular representation could then be computed using these results. If we consider the direct numerical decomposition of representations of $SU(2n)$ from chapter 4 the largest representations for which the calculation was possible were for $n = 2$ and $|\mathbf{f}| = 6$. While this could have been improved with more efficient algorithms the fundamental problem was that the dimension of the matrices grew as $(2n)^{|\mathbf{f}|}$ while the number of terms in the symmetriser also grew as approximately $|\mathbf{f}|!$. Therefore the analytic results provide a significant computational advantage.

As we saw previously for $\kappa \neq I$ A_κ is small as most contributions cancel. We can give a qualitative argument to show that this is inconsistent with all spin s multiplets transforming according to the same irreducible representation S_n. Each of the integrals A_κ is independent of the representation λ of S_n used to define $B^{s\lambda}$. Taking the set of solutions of the integrals A_κ we can regard them as a character of the symmetric group which is then decomposed into irreducible representations λ. With this picture we can consider what A_κ would look like if all multiplets with spin s were to transform according to the representation λ' of the symmetric group.

In this case we would require

$$A_\kappa = p\chi^{\lambda'}_{S_n}(\kappa) \tag{5.173}$$

where p is an integer. Let us consider the case when the dimension of the subspace with spin s is large, ie A_I is large. In this case p will also be large. However for $\kappa \neq I$ equation (5.173) will not agree with the formulae (5.171) and (5.172) where A_κ is small. We see that for typical representations of $SU(2n)$ multiplets with spin s will transform according to different representations of S_n.

Chapter 6

Conclusions

In chapter 3 we tasked ourselves with discovering the relation between spin and the statistics of particles in a position-dependent spin basis constructed using a general representation of $SU(2n)$. When we looked at the case of two particles with spin in chapter 4 we saw the first difficulty with constructing a general spin-statistics connection. The subspace W of vectors available to construct the position-dependent spin basis is formed from $s \otimes s$ multiplets with multiple values of spin. In the Berry-Robbins construction the spin-statistics connection involves the natural association between the completely symmetric representations of $SU(4)$, labelled by Young tableau with a single row, and the unique $s \otimes s$ multiplet that the representation contains. In the general case this type of relationship is not possible as fixing the representation of $SU(4)$ does not fix the value of spin. We did however notice that the representation of $SU(4)$ does determine whether the spin of the multiplets which make up W is integer or half integer.

Turning to the exchange sign we saw that in representations which contain several multiplets with the same spin s half (± 1 if the number of s multiplets is odd) of the multiplets will transform according to each irreducible representation of S_2. For these representations of $SU(4)$ there can be no clear spin-statistics connection as specifying the spin along with the representation is still insufficient to determine the exchange sign.

The decomposition of $SU(2n)$ using the characters of the physical representations of $\mathcal{H}$ shows that quantum mechanics on a position-dependent basis generated by a general representation of $SU(2n)$ admits parastatistics. We saw that, at least for spin-1/2, representations exist which display all types of parastatistics. An open question is whether there exist representations containing all forms of parastatistics for any spin. For any spin s and n particles position-dependent bases can be constructed which produce either bosonic or fermionic statistics. For example the representation of $SU(2n)$ labelled by a tableau with one row and $2sn$ boxes exhibits bosonic or fermionic statistics depending on whether s integer or half integer. If instead we select the representation of $SU(2n)$ labelled by a tableau $((2s+1)n, 1^n)$ this will also contain a single spin s multiplet. However the symmetry conditions of the representation, specifically the antisymmetry introduced in the first column, will contribute an extra sign change under the exchange of any two particles. In this representation half integer spin particles will be bosons and those with integer spin fermions. In general representations of $SU(2n)$ contain several multiplets with spin s which transform according to different irreducible representations of S_n. Typically such a representation will also contain subspaces with many different spins which can be used to generate the position-dependent basis. The full decomposition of a representation of $SU(2n)$ into the physical representations of $\mathcal{H}$ is given in terms of generalised Littlewood-Richardson and Clebsch-Gordan coefficients.

So what are the implications of these results for the spin-statistics connection? To include such a relationship what we require is a rule for selecting a position-dependent spin basis given a value of spin. If the rule is to associate a representation of $SU(2n)$ to each spin then the chosen representation of $SU(2n)$ must contain unique representations of $B^{s\lambda}$ when restricted to $\mathcal{H}$. In order for a representation $\mathbf{f}$ of $SU(2n)$ to fix s there can only be a single way of multiplying n tableau $\mathbf{s}$ and obtaining $\mathbf{f}$. The simplest strategy for selecting such a set of representations is to take the set of symmetric representations.

We see that the original scheme of Berry and Robbins is the most natural systematic approach to choosing a set of representations with which to construct the position-dependent spin bases that is afforded by the representations of $SU(2n)$. This does not amount to a spin-statistics theorem, as Berry and Robbins have already pointed out there still needs to be a convincing physical justification for insisting that spin vectors behave according only to these representations. We have however seen that extending the construction to general representations of $SU(2n)$ does not in general provide equally valid schemes for establishing a spin-statistics connection.

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